Book Review

Noeuds: Genèse d’une Théorie Mathématique

Reviewed by Alain Goriely

Noeuds: Genèse d’une Théorie Mathématique
Alexei Sossinsky
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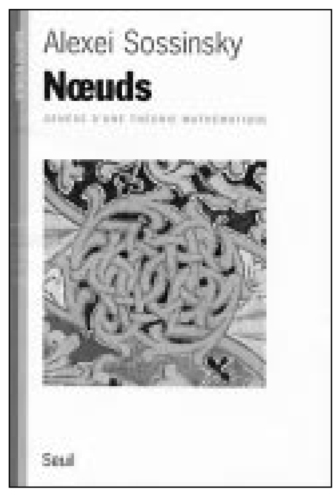

Knots are everywhere: they are in our shoelaces, in engineers’ cables, in sailors’ ropes, in the braids in our hair, and in the Olympic rings. On a minuscule scale they are in polymers and DNA molecules, and on a gigantic scale they are in magnetic vortex tubes in the solar corona. The mathematical theory of knots—the study of closed curves in three-dimensional space—is one of those rare theories that appeals to the pure, the applied, and the recreational mathematician. The study of mathematical knots, with its surprising tentacular connections to algebra, group theory, and statistical and quantum mechanics, is nowadays a central theme of mathematical research. At the same time, much like in number theory, some elementary open problems in knot theory can be explained to mathematically shy minds within minutes and lead to infinite variations of games and conjectures—a must at dinner parties. As a result, knot theory in the last decade has become a major mathematical area with its own journal and a favorite topic of coverage in popular science magazines. The field of knot books will soon be overcrowded, and there are to date many excellent ones on the market; ordered from the sandbox level to high priest revelations, my personal favorites are: *The Knots Puzzle Book* (a charming little book by H. McLeay), *The Knot Book* (for serious recreationers to apprentices, by C. C. Adams), *Knots and Surfaces* (where things become serious, by N. D. Gilbert and T. Porter), and finally *Knots and Physics* (at once obscure and illuminating, this book by L. H. Kauffman will keep the reader thinking for a while).

Is there a real need for yet another introductory knot book? There is always room for another book on my shelves, but the book under review is actually much less and much more than another book on knots. *Noeuds: Genèse d’une Théorie Mathématique* (*Knots: Genesis of a Mathematical Theory*) (pronounce the “oe” of “*Noeuds*” as in “oeuf” or “deux”, or else ask a French friend) is written by Alexei Sossinsky of the University of Moscow, a specialist in knot theory. Yes, this book is in French, but the conversational, non-Bourbaki style makes it accessible to a wide audience. I would not, however, recommend it for readers who want to learn French and knot theory in the same evening. Supposedly written for young readers “qui ont la bosse des math” (“who have an attraction [literally, a ‘hump’] to math”), *Noeuds* can be read at different levels, and for the first half the book does not require any knowledge of college mathematics. The last three chapters, however, will probably please college students and

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specialists alike. *Noeuds* covers so much material within 148 short pages that there is no room for an axiomatic construction, pages of exercises, or technical proofs. With this deliberate choice, many important topics such as colorability, knot groups, and knot volumes are not covered, but very advanced concepts such as Vassiliev invariants and quantum field theory are touched upon. Despite the lack of rigor, Sossinsky always points out the need for rigor in proving the theorems he so skillfully sketches. In many instances he first challenges the reader to find the proof, then gives a convincing argument, and finally points out where a plausible argument might fail and how it should be secured by rigor.

The book is organized historically into eight chapters, which can be read almost independently. Each of the first seven chapters reports on one main breakthrough in knot theory and revolves around a major scientific figure. Sossinsky starts with the atomic models of Lord Kelvin in 1860 and progresses to braids and knots (Alexander, 1923), Reidemeister planar diagrams (1928), knot arithmetic (Schubert, 1949), knot polynomials (Conway, 1973), and spin models and Kauffman brackets (1987), and rather ambitiously ends with the Vassiliev invariants (1990). The last chapter contains the author's own predictions and suggestions for future research.

Sossinsky's view of mathematical history is clearly one of singular personal achievements by brilliant individuals rather than one of incremental buildup of unostentatious details. However, the book is not a celebration of mathematical geniuses who industriously carry out the necessary steps of a God-given "program" (a rather fashionable notion in the modern world of preplanned grants and collaborative research). Rather, Sossinsky presents the history of knot theory as a history of grandiose failures, unjustified hopes, leaps of faith, and unprovable conjectures that each led to surprising new discoveries and unexpected connections with other theories. Doing so, Sossinsky adopts a psychological view of mathematics: a mature mathematical theory grows through its failures and unfulfilled desires.

The birth of knot theory itself is the result of a major scientific flop: the attempt by Lord Kelvin to model atoms by knots, which led Tait to establish the first table of alternating knots (an alternating knot is one whose planar projection has crossing points that alternate between over and under as one follows the knot in a given direction). The failures of both Alexander and Reidemeister to classify knots through, respectively, their equivalence with braids and transformations of their planar projections also led to major discoveries, and their works are nowadays central tools of knot theory. For instance, the three Reidemeister moves are local transformations of a knot projection that leave the knot type unchanged. Roughly speaking, two knots are equivalent if they can be continuously deformed into one another without letting the "strings" they are made of pass through each other; the knot type is then the class of all such equivalent knots. These transformations can be applied to the projection of a knot to simplify it or show its equivalence to another knot. The classical way to show that a quantity associated to a knot type is invariant is simply to demonstrate that the quantity is invariant under the Reidemeister moves, usually a manageable task. Knot invariants can thus be used to distinguish different knots. Unfortunately, no known invariants can actually distinguish all knots. The quest for knot invariants and their unlikely sources and applications is another major chapter of knot history (and the subject of three chapters of *Noeuds*). In this narrative of unexpected progress, Sossinsky succeeds remarkably well in showing the seemingly random paths that scientific thoughts follow. In this regard, *Noeuds* is much more than a book about knots; it presents knot theory as a showcase for mathematical construction.

Noeuds also succeeds in opening a window on individual mathematical reasoning. How do mathematicians obtain significant new results? Some mathematicians would like us to believe that there is a logical thought process to obtain new results, that the theory "calls for" a given approach. This view, reinforced by the often impersonal style of mathematical writings, seems to give an ethereal quality to their findings but at the same time reduces the creative role of mathematicians to that of pen pushers and transcribers of preexisting truths. Sossinsky demystifies this view by deconstructing one of the most important tools of knot theory: the Kauffman bracket. Rather than starting from its definition, Sossinsky first describes integrable statistical models and how, by analogy, planar models of interacting particles on a lattice (such as the celebrated Ising and Potts models) look like knots. It is then tempting to attach to a given knot projection a quantity, the Kauffman bracket, that plays the same role as the partition function of statistical mechanics (which can be used to compute the total energy of systems of interacting particles, the probability of finding a system in a given configuration, and the possible transition from different states). The bracket takes the form of a sum over all the possible "states" of a knot. The first step in the discovery process here is the analogy with a completely different scientific field. While this approach is not uncommon in mathematics, it is not usual. However, one could still argue that, by analogy with statistical mechanics, the analysis of knot diagrams naturally "calls for" the definition of a partition function. Despite this remarkable analogy, the function created by Kauffman has no intrinsic physical meaning, and the question remains, why did

Kauffman choose this particular form of partition function, which turned out to be central in subsequent analyses? To this Sossinsky answers: "Sans entrer dans les détails, je dirais simplement qu'il l'a trouvée en tâtonnant et en raisonnant à rebours, c'est-à-dire à partir du résultat qu'il cherchait à obtenir." ("Without going into detail, I would simply say that he found the bracket by backward reasoning and trial and error, that is, by starting from the result he wanted to obtain.") The secret is in the open: there is no a priori natural path of discovery; solecistic reasoning, intuitive leaps, and unfruitful attempts might be the real key to success.

Noeuds is also a very personal account of the author's feelings about mathematics. When describing the property of the Kauffman brackets and their implications for knot theory, Sossinsky becomes so ecstatic over the beauty of the results that he has to stop his exposition and share his euphoria with his reader:

> l'émotion que resent un mathématicien quand il rencontre (ou découvre) quelque chose de semblable est proche de celle qu'éprouve un amateur d'art lorsque son regard s'arrête sur la *Création* de Michel-Ange dans la chapelle Sixtine…. Ou encore de l'euphorie que doit resentir le chef d'orchestre quand tous les musiciens et les choristes, dans un même élan qu'il génère et qu'il domine, reprennent ≪l'hymne à la joie≫ à la fin du quatrième mouvement de la *Neuvième* de Beethoven….
>
> (The emotion that a mathematician experiences when he encounters (or discovers) something similar is close to what the art lover feels when he first looks at the *Creation* of Michaelangelo in the Sistine Chapel…. Or better yet, the euphoria that the conductor must experience when all the musicians and the choir, in the same breath that he instills and controls, repeats the "Ode to Joy" at the end of the fourth movement of Beethoven's *Ninth*….)

Sossinsky's exhilarated frankness will undoubtedly resonate with many mathematicians' own experiences.

Popular mathematical writers usually entice readers by stressing the recreational aspects of mathematics. Puzzles and games challenge the ingenuity of analytical minds and show that science can be fun. In all truth, fun is important in mathematics; it keeps mathematicians smiling through the day. But for many it is the inner beauty of an intellectual construction that represents the main appeal of science and transforms a profession into a passion or even an obsession. Sossinsky demonstrates that mathematics is more than fun and recreational; it is also beautiful, interesting, and important. The importance of a theory can be measured intrinsically by a common consensus. For instance, Schubert's theorem states that, much like numbers, every knot can be decomposed unequivocally into prime knots (the composition of two knots is the operation of cutting open each knot and gluing the loose ends together in pairs to obtain a new knot; a knot is prime if it cannot be decomposed into two nontrivial knots). Sossinsky weighs the importance of such a result: "Le théorème de Schubert n'a pas besoin de corollaires, c'est de l'art pour l'art mathématique de la plus haute volée" ("Schubert's theorem does not need corollaries; it is mathematical art for art's sake of the most exalted kind").

A mathematical theory can also be important extrinsically through its applications to other scientific fields and connections with other mathematical theories. Although there is not room in the book for many applications, the author succeeds in demonstrating the power of an abstract theory in a concrete scientific problem. In Conway's theory, a surgical operation is performed on knots. The operation consists in cutting a strand to let another one pass through before regluing the strands together; this operation then changes the knot type and the related invariants. The same operation is performed on DNA molecules by specialized enzymes called *topo-isomerases*. These enzymes in turn are part of the most important genetic processes, such as replication, transcription, and recombination. The *topo* in topo-isomerases stands for topology. Ironically, the word *topology* was coined by the German mathematician Listing to describe the geometry of plants and is now the main contribution of mathematics to biological terminology.

The last chapter focuses on the future of knot theory. Close to stream-of-consciousness writing, this chapter explores without much explanation all the possible connections and strange parallels between knot theory and mathematical physics. Rather than giving a standard list of open problems and homework, Sossinsky hints that coincidences between different fields are more than just coincidences: for example, the Yang-Baxter equation in statistical physics, the Feynman diagrams in quantum field theory, and Kontsevich integrals for the Vassiliev invariants. Finally, the chapter closes by concluding that maybe Lord Kelvin's idea of knots as a unifying physics principle might not be so bad after all.

The book has one drawback: many equations in the text are inconsistently labeled (especially in Chapter 6), and many references are missing from the final bibliography. While this does not affect a casual reading, the text would certainly benefit from a revised edition (an English translation would also be more than welcome).

In conclusion, I warmly recommend *Noeuds*. The many digressions and personal thoughts of the author make the reading truly enjoyable. It is probably the first mathematically oriented book that I have read cover to cover without putting it down (oxymoronically, my first mathematical page-turner). It leaves the reader with the magical feeling of having peeked at a theory beyond normal reach and the optimistic view that the discoveries to come will bring as many surprises and marvels as the ones found in this book.

References

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