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Necessary and Sufficient Conditions for Finite Time Singularities in Ordinary Differential Equations

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This paper gives a theorem which provides necessary and sufficient conditions for the existence of a real finite value of the independent variable at which the general solution of a certain class of ordinary differential equations diverges to infinity. This class is a large subset of the set of all autonomous, non-linear polynomial ordinary differential equation. These conditions involve the asymptotic form of the local series representation for the general solutions around the singularities and can be checked algorithmically. © 2000 Academic Press

Key Words: singularity analysis; finite time singularities.

1. INTRODUCTION

It is well known that systems of nonlinear ordinary differential equations can exhibit finite-time singularities. These are the values of the independent variable (refer to as *the time*), where the norm of the solution diverges. The simplest example is the one dimensional equation:

$$\frac{dx}{dt} \equiv \dot{x} = ax^3 \tag{1}$$

whose general solution (for positive initial conditions) is:

$$x(t) = \frac{1}{\sqrt{x(0)^{-2} - 2at}} \tag{2}$$

Depending on the value of a, the solution has a finite-time singularity ($a > 0$ with singularity $t_* = 1/(2ax(0)^2)$) or is bounded for all time ($a < 0$).

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This simple equation as yet another solution, that is valid close enough to the singularity. Indeed, it is easy to check that

$$x(t) = \frac{t^{-1/2}}{\sqrt{-2a}} \tag{3}$$

is another (particular) solution. The interesting feature of this solution is that it is complex for $a < 0$ and $t < 0$ and real if $a > 0$ and $t < 0$. Therefore a natural question to ask is whether this property holds in more general settings. Indeed, the particular solution around the singularities can be computed algorithmically whereas the general solution can not be obtained in general.

This elementary observation seems to indicate that there is a simple connection between the reality of the leading coefficient and the occurrence of blow-up and leads naturally to the following questions:

1. Does a real leading coefficient ensure the occurrence of blow-up?
2. Does a real time singularity imply that the leading coefficient of one of the asymptotic series is real?

It is the purpose of this paper to answer these questions affirmatively.

More specifically, we consider here ODEs which are part of a large subset (to be defined later) of the class of systems of autonomous nonlinear polynomial ODEs:

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n, \tag{4}$$

where $\dot{x} = dx/dt$.

The main problem in showing the existence of a singularity for the general solution of a system of ODEs is that the singularities' locations change with the initial conditions. Indeed, *movable singularities* are the only possible type of singularities for autonomous polynomial systems, but they are, in general, complex valued and therefore do not always occur in the *real time* dynamics. The second problem is that most of the systems do not exhibit real singularities for all initial conditions. Therefore, we have to formulate the problem of real time blow-up in the following way: *Find necessary and sufficient conditions for the existence of an open set of initial conditions such that all solutions based on these initial conditions exhibit finite real time singularities.* These conditions do not guarantee that the solutions are bounded, indeed boundedness imposes that the solutions do not grow indefinitely in time but are bounded in a region of phase-space. The linear system $\dot{x} = x$, $x \in \mathbb{R}$ does not have a bounded general solution, however it does not exhibit any finite time singularity (real or complex).

Nevertheless, the absence of finite time blow-up is a necessary condition to prove the boundedness of solutions.

In order to find these conditions we will analyze the asymptotic form of the general solutions around the movable singularities. This type of analysis is based on the so-called *singularity analysis*. It is usually used to prove the integrability of ODEs [1] (or PDEs [2]). In this case, one seeks to find necessary conditions for the Painlevé property by requiring that the local solutions around the movable singularities are Laurent series. It provides a straightforward and algorithmic test for integrability. If the local series involve logarithmic terms, the singularity analysis can be used to show the non-existence of first integrals [3] or, with additional assumptions, to compute the splitting of separatrices in perturbed integrable systems [4]. One of the difficulties related to the singularity analysis is the multiplicity of asymptotic solutions around the singularities. Indeed, as we will show here, different asymptotic solutions can be found, and these different solutions correspond either to the asymptotic solutions of a general solution around different singularities or the asymptotic solutions of different type of solutions (particular solutions, similarity invariant solutions, etc.). Therefore, one has to identify which expansions are related to the general solutions. For those which are, we show here that these series are local expansions around a real time singularity if and only if all the coefficients in the series are real. Due to the particular structure of the series, this amounts to showing that the leading coefficient is real.

The structure of this paper is as follows: In Section II, the different notions relative to singularity analysis, finite-time blow-up and the formal existence of local solutions around the singularities are introduced. In Section III, the main theorem relating the existence of finite time blow-up to the reality of the coefficients in the local series is given and proved. In Section IV, secondary results on the position of blow-up, the absence of singularities and the relation to first integrals are given. Finally, Section V shows some examples of our general results. Section VI is Discussions and Conclusions.

2. SETUP OF THE GENERAL PROBLEM

Consider a system of n first order ODEs:

$$\dot{x} = f(x), \tag{5}$$

where $x \in \mathbb{R}^n$ and $f_i(x)$ are polynomial functions of x with real coefficients.

The *general solution* of (5) is a solution that contains n arbitrary constants. We are interested in the general solutions because they describe the time-evolution of the system for arbitrary initial data. By contrast, the *particular solutions* contain less than n arbitrary constants and do not describe the evolution of arbitrary initial data, but rather the evolution of restricted subsets of initial data and/or envelope solutions. The general solution will be denoted $x=x(t; C_1, ..., C_n)$. In the same way, the solution based on the initial condition $x(0)=x_0$ will be $x=x(t; x_0)$.

A solution will exhibit *finite time blow-up* if there exist $t_* \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$ such that for all $M \in \mathbb{R}$, there exists an $\varepsilon > 0$ satisfying

$$|t-t_*| < \varepsilon \Rightarrow \|x(t; x_0)\| > M, \tag{6}$$

where $\|\cdot\|$ is any l^p norm.

Equivalently, we will use "$\lim_{t \to t_*} \|x(t, x_0)\| \to \infty$" to denote such a finite time blow-up.

In order to study finite time blow-up in the solutions of differential equations, the solutions need to be analyzed locally around the singularity. The singularity analysis (also known as the Painlevé–Kowalevskaya test) is a well studied field for integrable systems. It relates the existence of local Laurent series around the singularities to the global property of integrability (see for instance [5, 3]). More precisely, a system of ODEs is said to have the Painlevé property if the general solution is meromorphic. The Painlevé test provides necessary conditions for the Painlevé property by requiring that all local solutions around the singularities can be expanded in Laurent series. In general most of the systems of ODEs are not integrable and their solutions cannot be locally expanded in Laurent series. However, it has been proven that the analysis of the local expansions can still provide valuable insight into the real-time dynamics of the solutions [9]. We now sketch the singularity analysis and the different types of solutions that can be found around the singularities. Our goal is to build local solutions around the singularities in order to study the different aspects relevant to finite time blow-up. The local solutions considered here are the so-called *Psi-series* [6], defined by:

$$x = \Psi(\alpha, p, t) \equiv \alpha \tau^p \left[\alpha + \sum_{j=1}^{\infty} a_j \tau^{j/q} \right], \tag{7}$$

where $\tau = (t_* - t)$, $p \in \mathbb{Q}^n$, $q \in \mathbb{N}$ and a_j is a polynomial in $\log(t_* - t)$ of degree $N_j \leqslant j$.

The different characteristics of these series can be found algorithmically by the following procedure:

Step 1. The first step of the singularity analysis consists in finding all the truncations $\hat{f}$ of the vector field f

$$\dot{x} = f(x) = \hat{f}(x) + \check{f}(x), \tag{8}$$

such that the *dominant behavior* $x = \alpha\tau^p$, $\alpha \in \mathbb{C}_0^n$ is an exact solution of the truncated system

$$\dot{x} = \hat{f}(x), \tag{9}$$

where $p \in \mathbb{Q}^n$ with at least one negative component.

In order for $x = \alpha\tau^p$ to be the dominant behavior, it is also required that $\check{f}(x)$ is *not dominant*, that is, at the singularity:

$$\check{f}(\alpha(t_* - t)^p) \underset{t \to t_*}{\sim} \check{\alpha}(t_* - t)^{p + \check{p} - 1}, \tag{10}$$

with $\check{p} \in \mathbb{Q}^n$ and each $\check{p}_i > 0$.

Each truncation defines a *balance* (α, p). Every balance corresponds to the first term $\alpha\tau^p$ in an expansion around movable singularities and different balances correspond to different expansions around different singularities. One of the difficulties of the singularity analysis is to keep track of all the different balances a system may exhibit. In order to check if this series is an expansion of the *general solution* one has to find the number of arbitrary constants in the series defined by a given balance. To do so, we have to compute the so-called resonances of the series.

Step 2. The second step is the computation of the *resonances*. Each balance defines a new set of resonances. These resonances are related to the indices j of the coefficients a_j in the Psi-series (7) at which arbitrary constants first appear (specifically, j/q is a resonance if a new arbitrary constant is introduced in the computation of a_j for the series (7)). It is a standard matter [8] to show that these resonances are given by the eigenvalues of the matrix $\mathscr{R}$:

$$\mathscr{R} = -D\hat{f}(\alpha) - \operatorname{diag}(p), \tag{11}$$

where $D\hat{f}(\alpha)$ is the Jacobian matrix evaluated on α.

The resonances are labeled r_i, $i = 1, ..., n$ with $r_1 = -1$. In view of the form (7), the only resonances allowed here are of the form $r_i = \rho_i/q$, $\rho_i \in \mathbb{Z}$ $\forall i = 1, ..., n$, where $q \in \mathbb{N}$.

A *general solution* is a formal solution $x = \Psi(\alpha, p, t)$ with balance (α, p) such that $r_j \geqslant 0$ for all $j > 1$. That is, the Psi-series built on that balance contains $(n-1)$ arbitrary coefficients (the final arbitrary constants being the singularity position t_*).

Step 3. The third and last step of the singularity analysis consists is finding the explicit form of the different coefficients a_j. In general, these coefficients are vector valued polynomials in $\log(t-t_*)$ of degree $N_j \leqslant j$. These coefficients are computed by inserting the full Psi-series (7) in the original system (5) and by determining explicitly the recursion relation for the coefficients a_{jk} appearing in $a_j = \sum_{k=0}^{N_j} a_{jk} \log(t_* - t)^k$. The formal existence of these series is guaranteed by the following lemma, proven in the appendix:

LEMMA 1. *The series given in* (7) *is a general formal solution to the system* (5).

The important point to note here is that the polynomials a_j are functions of the different arbitrary coefficients $(c_2, ..., c_n)$ entering at each resonance in the following way:

$$a_j = a_j(\alpha, c_2, ..., c_k), \tag{12}$$

where c_i is the arbitrary constant corresponding to the resonance r_i and $r_k \leqslant j/q$. By definition we take $c_1 = t_*$ as the arbitrary position of the singularity is known to be associated with the resonance $r_1 = -1$. We denote $c = (c_1, ..., c_n)$ and $\Psi(\alpha, p, t; c)$ as the series (7) with balance (α, p) and arbitrary constants c.

A recursion relation can be obtained to relate the reality of the arbitrary coefficients and the leading behavior to the reality of the coefficients a_{jk}:

LEMMA 2. *Let* $x = \Psi(\alpha, p, t; c_2, ..., c_m)$ *be a solution of* $\dot{x} = f(x)$ *around the singularity* t_* *containing* $(m-1)$ *arbitrary coefficients. If* $\alpha \in \mathbb{R}^n$, *and* $c_i \in \mathbb{R}\ \forall\ i = 2, ..., m$ *then* $a_{jk} \in \mathbb{R}^n\ \forall (j, k)$

A proof of this lemma can also be found in the appendix.

Different special cases are of interest: A necessary conditions for the Painlevé property is that for all balances (α, p) we have $p, \check{p} \in \mathbb{N}^n$, $q = 1$ and a_j constant for all j. The system is then said to pass the Painlevé test and, as stated in the introduction, it strongly suggests that the system is actually integrable ([7]). If $q \neq 1$ but a_j is constant for all j then the system has the weak-Painlevé property and can (in some cases) be shown to be integrable (see [5, 8]).

In order for these series solutions to exist, their convergence (in a punctured disk around the singularity) has to be asserted. This is covered by the following assumption:

ASSUMPTION 1. *There exists a non-empty closed connected set* $\bar{\mathbf{C}} \in \mathbb{C}^n$ *such that* $\forall c \in \bar{\mathbf{C}}$, *the solutions* $x = \Psi(\alpha, p, t; c)$ *of* (5) *are convergent in an open punctured disk* D_{t_*} *around the singularity* t_*.

We shall denote the radius of convergence of $\Psi(\alpha, p, t; c)$ by δ_c. The nature of the set $\bar{\mathbf{C}}$ ensures that

$$\gamma = \min\{1, \inf_{c \in \bar{\mathbf{C}}} \{\delta_c\}\}, \tag{13}$$

exists and is strictly positive.

In the case where the Psi-series reduce to Puiseux series (i.e. without logarithmic terms), the convergence of these series has been proven in [9, 10]. In the general case, recent general results on singular analysis for PDEs by Kichenassamy and co-workers [11, 12, 13] strongly suggest that that the Psi-series are convergent in general as has been successfully demonstrated on many specific examples [14, 15, 16, 17]. However, in the absence of a well-defined rigorous proof, we leave here the convergence of the Psi-series as an assumption.

3. MAIN THEOREM

We now show that the leading behavior of the series (7) is real, if and only if the solution exhibit finite time blow-up on an open set of initial conditions.

DEFINITION. Let F_n be the set of all n-dimensional real nonlinear polynomial vector fields $f(x)$ such that the system $\dot{x} = f(x)$ has a convergent (in the sense of Assumption 1) general solution $x = \Psi(\alpha, p, t; c)$ with $p \in \mathbb{Q}^n$ and $\mathrm{Spec}(R) \backslash \{-1\} \in \mathbb{Q}_+^{n-1}$.

THEOREM. *Consider the system* $\dot{x} = f(x)$ *where* $f \in \mathrm{F}_n$, $x \in \mathbb{R}^n$. *Then the two following statements are equivalent*:

(a) *There exists an open set of initial conditions* $\mathbf{X}_0 \subseteq \mathbb{R}^n$ *such that for all* $x_0 \in \mathbf{X}_0$, *there exists a* $t_* \in \mathbb{R}$ *for which* $\lim_{t \to t_*} \|x(t, x_0)\| \to \infty$.

(b) *There exists a general solution* $x = \Psi(\alpha, p, t; c)$ *with* $\alpha \in \mathbb{R}^n$.

Strategy of the proof. We split the proof in two parts.

First, we prove that (a) $\Rightarrow$ (b), that is, assuming the existence of real singularities for an open set of initial conditions, the leading behavior α must be real-valued.

Second, we show that (b) $\Rightarrow$ (a), that is the existence of general solutions with real leading behavior is enough to ensure the existence of finite time blow-up on an open set of initial conditions. The main idea is to use the local representation of the series ($x = \Psi(\alpha, p, t; c)$) to build an open set of initial conditions. To do so, we show that there exists a homeomorphism

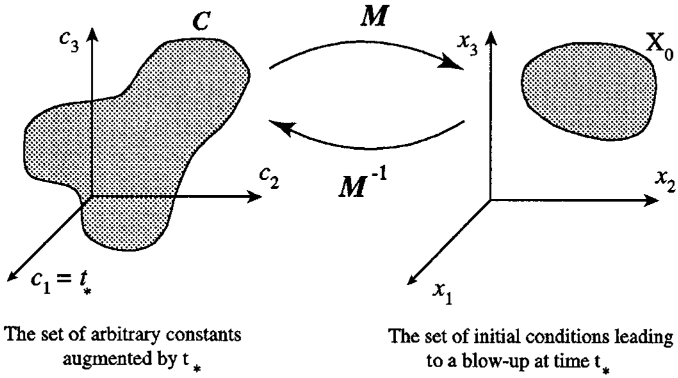

FIG. 1. The map M maps an open set of arbitrary coefficient to an open set of initial conditions leading to finite time blow-up.

$M: \mathbf{C} \to \mathbf{X}_0$, between an open set of arbitrary constants $\mathbf{C} \subset \mathbb{R}^n$ appearing in the general solutions and an open set of initial conditions $\mathbf{X}_0$ leading to finite time blow-up (see Fig. 1).

3.1. (a) $\Rightarrow$ (b)

Let $x_0 \in \mathbf{X}_0 \subset \mathbb{R}^n$. By hypothesis, there exists $t_* \in \mathbb{R}$ which is the blow-up time associated with x_0. For all $t \in]t_* - \gamma, t_*[$, $x(t; x_0)$ is real (since t is real). In this interval, we can use the representation of $x(t; x_0)$ provided by (7):

$$x(t; x_0) = \mathrm{Re}(x) = \tau^p \left(\mathrm{Re}(\alpha) + \sum_{j=1}^{\infty} \mathrm{Re}(a_j)\, \tau^{j/q} + \mathrm{Im}(\alpha) + \sum_{j=1}^{\infty} \mathrm{Im}(a_j)\, \tau^{j/q} \right). \tag{14}$$

This implies that

$$\mathrm{Im}(\alpha) + \sum_{j=1}^{\infty} \mathrm{Im}(a_j)\, \tau^{j/q} = 0, \tag{15}$$

for all $t \in]t_* - \gamma, t_*[$. This, however, implies that

$$\mathrm{Im}(\alpha) = \mathrm{Im}(a_j) = 0 \Rightarrow \alpha \in \mathbb{R}^n, \tag{16}$$

3.2. (a) $\Leftarrow$ (b)

By assumption, we can represent, locally around a movable singularity t_*, a solution of $\dot{x} = f(x)$, $f \in F_n$ by a series of the form $x = \Psi(\alpha, p, t; \mathbf{c})$ where $\alpha \in \mathbb{R}^n$. According to Lemma 2, we have:

$$\mathbf{c} \in \mathbb{R}^n, \qquad \alpha \in \mathbb{R}^n \Rightarrow \Psi(\alpha, p, t; \mathbf{c}) \in \mathbb{R}^n \quad \forall t \in]t_* - \gamma, t_*[. \tag{17}$$

Now choose an open set $\mathbf{C} \subset \bar{\mathbf{C}} \subset \mathbb{R}^n$, such that for all $\mathbf{c} \in \mathbf{C}$, the vector of arbitrary constants $\mathbf{c}$ satisfies $t_*^{min} < c_1 < t_*^{max}$ for any real numbers t_*^{min} and t_*^{max} satisfying $t_*^{max} - t_*^{min} \leqslant \gamma/4$ (note that c_1 is defined to be t_*.) Thus, for $\mathbf{c} \in \mathbf{C}$, the series $\Psi(\alpha, p, t; \mathbf{c})$ can be used to define a set of initial conditions leading to finite-time blow-up. Indeed, for $\mathbf{c} \in \mathbf{C}$, we can pick $t_0 = t_*^{min} - \gamma/4$ and define:

$$x_0 = x(t_0) = \Psi(\alpha, p, t_0; \mathbf{c}), \tag{18}$$

where (α, p) is a given balance corresponding to a general solution with $\alpha \in \mathbb{R}^n$. The solution $x(t)$ based on the initial condition x_0 will blow-up at $t_* =]t_0 + \gamma/4,\ t_0 + \gamma/2[$. By varying $\mathbf{c}$ in $\mathbf{C}$, we can define the set $\mathbf{X}_0$:

$$\mathbf{X}_0 = \left\{ x_0 = \Psi(\alpha, p, t_0; \mathbf{c}); \mathbf{c} \in \mathbf{C}, t_0 = t_*^{min} - \frac{\gamma}{4} \right\}. \tag{19}$$

Note that it may seem counter-intuitive that one can simply *choose* a real value of t_*. However, the original system is invariant under time shift, so it is not surprising that the value t_* can range over any value, as the initial condition is defined at a time t_0 relative to the range of allowed values of t_*. Furthermore, one can always choose real values for $\mathbf{c}$ and then evaluate the resulting Psi-series at an appropriately close value of t_0. However, doing so in general results in complex-valued initial conditions. It is only for the case where $\alpha \in \mathbb{R}^n$ that choosing a real-valued set of arbitrary constants $\mathbf{c}$ (including $t_* = c_1 \in \mathbb{R}$) leads to real-valued initial conditions.

In order to show that $\mathbf{X}_0$ is an open set, we have to prove that the map M

$$M : \mathbf{C} \to \mathbf{X}_0, \tag{20}$$

is a homeomorphism. This in turn implies that M^{-1} is continuous and therefore that $\mathbf{X}_0 = M(\mathbf{C})$ is an open set.

Thus, choosing the set $\mathbf{C}$ to be real and open gives us that the set $\mathbf{X}_0$ of corresponding initial conditions is also real and open. We now prove that the map M is a homeomorphism, that is, it is (i) single-valued and one-to-one and (ii) continuous.

(i) *M* is single-valued and one-to-one

We consider two initial conditions $x_0 = x(t_0; c_k)$, $\tilde{x}_0 = x(t_0; \tilde{c}_k)$, $\in \mathbf{X}_0$ by considering $\mathbf{c}, \tilde{\mathbf{c}} \in \mathbf{C}$ in such a way that $c_k \neq \tilde{c}_k$ for $k > 1$ and $c_i = \tilde{c}_i$, $i \neq k$. Accordingly, we define $\tilde{x}(t) = x(t; \tilde{c}_k)$. Then from the Psi-series (7), we have:

$$x(t) - \tilde{x}(t) = (c_k - \tilde{c}_k)\, y_k \tau^{-p + r_k/q}(1 + O(\tau^{1/q})), \tag{21}$$

for all $t_0 \leqslant t < t_*$, where y_k denotes the normalized eignevector associated with the kth resonance r_k (see appendix). From the above equation, we get

$$x_0 = \tilde{x}_0 \Leftrightarrow x(t) = \tilde{x}(t) \Leftrightarrow c_k = \tilde{c}_k, \tag{22}$$

where the first correspondence is a direct consequence of the existence and uniqueness of the solutions away from $t = t_*$.

Next, we consider the case where $\tilde{c}_1 \neq c_1$ (that is $\tilde{t}_* \neq t_*$). Let $\tilde{x}_0 = x(t_0; \tilde{c}_1)$ while $x_0 = x(t_0; c_1)$ with $c_i = \tilde{c}_i$, $i > 1$. Let $x(t) = x(t; x_0)$ be the solution based at x_0. The following equality follows from the fact that the Psi-series are functions of $(t_* - t)$ only:

$$\Psi(\alpha, p, t; (c_1, c_2, ..., c_n)) = \Psi(\alpha, p, t + a; (c_1 + a, c_2, ..., c_n)) \qquad \forall a \in \mathbb{C}. \tag{23}$$

As a consequence, we have:

$$\begin{aligned}
\tilde{x}_0 &= \Psi(\alpha, p, t_0; \tilde{c}) \\
&= \Psi(\alpha, p, t_0; (\tilde{t}_*, c_2, ..., c_n)) \\
&= \Psi(\alpha, p, t_0 + (t_* - \tilde{t}_*); (t_*, c_2, ..., c_n)) \\
&= \Psi(\alpha, p, t_0 + (t_* - \tilde{t}_*); \mathbf{c})) \\
&= x(t_0 + (t_* - \tilde{t}_*); x_0).
\end{aligned} \tag{24}$$

Therefore, by the uniqueness of the solutions, we have $x_0 = \tilde{x}_0$ if and only if $t_* = \tilde{t}_*$ (the case where $t_* \neq \tilde{t}_*$ and $x_0 = \tilde{x}_0$ can only happen on periodic orbits which are excluded here since they cannot blow up in finite time). By the same token, due to to the continuity of the flow, the relation (24) guarantees that the map M is continuous in the variable c_1.

(ii) M is continuous.

By definition, the map M is continuous if

$$\forall \varepsilon > 0, \qquad \exists \eta_\varepsilon \ni \|\tilde{\mathbf{c}} - \mathbf{c}\| < \eta_\varepsilon \Rightarrow \|\tilde{\mathbf{x}}_0 - \mathbf{x}_0\| < \varepsilon, \tag{25}$$

where $\mathbf{c}, \tilde{\mathbf{c}} \in \mathbf{C}$, $\mathbf{x}_0, \tilde{\mathbf{x}}_0 \in \mathbf{X}_0$ and $\|\cdot\|$ is the infinity norm.

We have already shown that the map M is continuous with respect to its first argument (c_1). Now, let

$$\|\tilde{\mathbf{c}} - \mathbf{c}\| = |\tilde{c}_k - c_k|, \tag{26}$$

for one or more of the c_k $(k>1)$. Let $\beta = t_* - t_0$, then by definition $0<\beta \leqslant \gamma/2 \leqslant \frac{1}{2}$. Since the series $x_0 = \Psi(\alpha, p, t_0; \mathbf{c})$ and $\tilde{x}_0 = \Psi(\alpha, p, t_0; \tilde{\mathbf{c}})$ converge, so does the series for $x_0 - \tilde{x}_0$:

$$x_0 - \tilde{x}_0 = (c_k - \tilde{c}_k)\, y_k \beta^{-p+r_k} + \sum_{j=r_k+1}^{\infty} (a_j - \tilde{a}_j)\, \beta^{-p+j/q}, \tag{27}$$

where r_k is the kth resonance and $c_j = \tilde{c}_j\ \forall j<k$. (For maximum generality, we shall let r_k be the smallest non-negative resonance, so by assumption $r_k>0$.) Since this series converges, the tail can be made arbitrarily small, i.e. for any finite pair of values c_k and $\tilde{c}_k$, we have

$$\forall \nu>0, \qquad \exists N_{(c_k,\, \tilde{c}_k)} \in \mathbb{N} \ni \left| \sum_{j=N_{(c_k,\, \tilde{c}_k)}}^{\infty} (a_j - \tilde{a}_j)\, \beta^{-p+j/q} \right| < \nu. \tag{28}$$

Let $N = \sup_{c_k,\, \tilde{c}_k \in \mathbf{c}} \{N_{(c_k,\, \tilde{c}_k)}\}$.

From the polynomial recursion relations (12), it follows that $a_j = a_j(c_k)$ is continuous (see appendix). Similarly, $\tilde{a}_j$ is continuous in $\tilde{c}_k$ and so $(a_j - \tilde{a}_j)$ is a continuous function of both c_k and $\tilde{c}_k - c_k$. Therefore, for any fixed c_k, the following is true:

$$\forall \mu>0, \qquad \exists \eta_j > 0 \ni |\tilde{c}_k - c_k| < \eta_j \Rightarrow \|a_j - \tilde{a}_j\| < \mu \quad \text{for} \quad j<N. \tag{29}$$

Let $\eta = \inf_{j\in\{0,\, ...,\, N-1\}} \eta_j$ for a given μ. Choosing $|\tilde{c}_k - c_k| < \eta$ guarantees that

$$\begin{aligned} |\tilde{x}_0 - x_0| &\leqslant \left| \beta^{-p+r_k} \sum_{j=0}^{N-1} (\tilde{a}_j - a_j)\, \beta^{j/q} \right| + \nu \\ &\leqslant \left(\mu \beta^{-p+r_k} \sum_{j=0}^{\infty} \beta^{j/q} \right) + \nu \\ &\leqslant \mu \frac{\beta^{-p+r_k}}{1-\beta^{1/q}} + \nu. \end{aligned} \tag{30}$$

Letting $\nu = \varepsilon/2$ and $\mu = \varepsilon\, ((1-\beta^{1/q})/2\beta^{-p+r_k})$, we obtain

$$|\tilde{c}_k - c_k| < \eta \Rightarrow \|\tilde{x}_0 - x_0\| < \varepsilon. \tag{31}$$

(Note that η is a function of μ which is a function of ε). Thus, the map M is continuous in c_k for all $k = 1, ..., n$. ∎

4. SECONDARY RESULTS

4.1. *Absence of Singularities and Blow-up Regions*

As an obvious consequence of the theorem, the absence of real singularities can be tested:

COROLLARY 1. *The system* $\dot{x}=f(x)$, $f\in F_n$ *does not have finite time singularities if for all general solutions of the form* (5), $\mathrm{Im}(\alpha)\neq(0, ..., 0)$.

In general, the open set of initial conditions leading to a blow-up cannot be computed. However, the orthant in phase space (i.e. one of the 2^n regions of $\mathbb{R}^n$ defined by $\{\mathrm{sign}(x_i),\ i=1, ..., n\}$) where blow-up occurs can be readily obtained:

PROPOSITION 2. *The orthant in phase space in which blow-up occurs is the orthant of* $\alpha\in\mathbb{R}^n$.

Proof 2. When blow-up occurs, the leading behavior is dominant, therefore, for t close enough to the singularity we have:

$$x_i(t)=\alpha_i(t_*-t)^p+O((t_*-t)^{p+1/q}). \tag{32}$$

Therefore the sign of x_i is given by the sign of α_i. ∎

Note here that the orthant is defined including the border axes (for instance in two-dimensions (x_1, x_2), the first quadrant $\{+, +\}$ includes the semi-axes $x_1\geqslant 0$ and $x_2\geqslant 0$). This accounts for the case where some components of the p vector are positive.

4.2 *Finite Time Blow-up and First Integrals*

We now discuss the existence of finite time blow-up in the presence of first integrals. In some cases, polynomial systems $\dot{x}=f(x)$ can have *first integrals*, that is function $J=J(x, t)$ such that $\dot{J}=\nabla J\cdot f+\partial_t J=0$. These first integrals are constant on any solutions of the system. In some instances, these conserved quantities can be used to prove directly the absence of finite time blow-up. For instance, if a two-dimensional system has a first integral $J=x_1^2+x_2^2$, it is straightforward to see that there is no possibility of finite-time blow-up $(J=x_{01}^2+x_{02}^2=x_1^2+x_2^2\in\mathbb{R}\Rightarrow x_1, x_2\in\mathbb{R}\ \forall t)$. If, however, $J=x_1^2-x_2^2$, then blow-up cannot be ruled out as the solutions may go to infinity in such a way that the difference of the squares remains constant. It is therefore straightforward to prove that:

PROPOSITION 3. *Let $J = J(x, t)$ be a first integral for the system $\dot{x} = f(x)$, $x \in \mathbb{R}^n$. If the level sets of J are compact then there is no finite time-blow-up.*

How is this well-known result related to Corollary 1 on the absence of blow-up? If $J = J(x, t)$ is a first integral for the system $\dot{x} = f(x)$, then there exists a first integral $\hat{J} = \hat{J}(x)$ for the system $\dot{x} = \hat{f}(x)$, where $\hat{f}(x)$ is, as before, a dominant part of the vector field. That is, the system $\dot{x} = \hat{f}(x)$ has an exact solution $x = \alpha\tau^p$ and $f = \hat{f} + \check{f}$; $J(x, t) = (\hat{J}(x) + \check{J}(x))\, g(t)$. In other words, the dominant part of the first integral is a first integral of the dominant part of the vector field (See [3] for further details). Since the first integral $\hat{J}$ is constant on all solutions, it is constant on the particular solution $x = \alpha\tau^p$, therefore $\hat{J}(\alpha\tau^p) = \hat{J}(\alpha)\, \tau^d = 0 \Rightarrow \hat{J}(\alpha) = 0$. However, if $J(x, t)$ is of definite sign, so is $\hat{J}(x, t)$ and therefore the relation $\hat{J}(\alpha) = 0$ cannot be satisfied if $\alpha \in \mathbb{R}^n$, which proves that Proposition 3 is a direct consequence of Corollary 1. So, the fact that $\hat{J}$ is of definite sign implies that the corresponding balance (α, p) is such that $\mathrm{Im}(\alpha) \neq 0$. Moreover, we can propose an upgraded version of Proposition 3:

PROPOSITION 4. *Let $\hat{J} = \hat{J}(x)$ be a first integral of a dominant part of the system $\dot{x} = f(x)$, $x \in \mathbb{R}^n$. If the level sets of $\hat{J}$ are compact then there is no finite time-blow-up.*

See Section V.2 for an illustration of Proposition 4.

5. APPLICATIONS

5.1. *A Simple Example*

We consider the system

$$\dot{x}_1 = x_1(a + bx_2), \tag{33.a}$$

$$\dot{x}_2 = cx_1^2 + dx_2, \tag{33.b}$$

with $b > 0$, $c > 0$. This system arises from the reduction of a semilinear parabolic PDE [18]. The existence of finite time blow-up for this system is used to prove the finite time blow-up of the PDE. We show how our theorem can be used to immediately determine the existence of finite time blow-up for this system.

The first step of the analysis is to determine the different balances, that is the different possible dominant truncations of the vector field. In this case, we find two balances both corresponding to the truncation:

$$f=\begin{pmatrix} x_1(a+bx_2) \\ cx_1^2+dx_2 \end{pmatrix}, \qquad \hat{f}=\begin{pmatrix} bx_1x_2 \\ cx_1^2 \end{pmatrix}, \quad \check{f}=\begin{pmatrix} ax_1 \\ dx_2 \end{pmatrix}, \tag{34}$$

and $p=(-1,-1)$. That is, the system $\dot{x}=\hat{f}(x)$ has two exact solutions of the form $x=\alpha\tau^p$ with $\alpha=(\pm 1/\sqrt{bc}, 1/c)$. It is easy to check, using (11), that the resonances are, in both cases, $r=-1$ and $r=2$. These two balances define Psi-series solutions where logarithmic terms enter as coefficients of the term $\tau^{p+r}=\tau^1$. Therefore, we can apply our main theorem and state that for all b, c such that $bc>0$ the system (33) exhibits finite time blow-up, that is there exist open sets of initial conditions in $\mathbb{R}^2$ such that all solutions based on this set blow-up in finite-time. Moreover, the blow-up occurs both in the first ($\{+,+\}$) and fourth ($\{-,+\}$) quadrants (see Proposition 2).

5.2. *Absence of Finite-time Singularity for the Lorenz System*

The Lorenz system is ubiquitous in dynamical system theory [19, 20] integrability theories [21, 22] and singularity analysis theory [23, 24, 3]. The boundedness of its solution was proved in [25]. The system reads:

$$\dot{x}=\sigma(y-x) \tag{35.a}$$

$$\dot{y}=\rho x-y-xz \tag{35.b}$$

$$\dot{z}=xy-\beta z, \tag{35.c}$$

where $x, y, z, \sigma, \beta, \rho \in \mathbb{R}$.

The Lorenz system has only two balances characterized by the leading behavior $p=(-1,-2,-2)$ and $\alpha=(\pm 2i, \mp 2i/\sigma, -2/\sigma)$. Both balances are associated with the truncation:

$$\hat{f}=\begin{pmatrix} \sigma y \\ xz \\ xy \end{pmatrix}, \qquad \check{f}=\begin{pmatrix} \sigma x \\ \rho x-y \\ -\beta z \end{pmatrix}, \tag{36}$$

These balances define the first terms of the Psi-series characterizing the local solutions around the singularities. The resonances are $\mathrm{Spec}(R)=\{-1,2,4\}$ which shows that the Psi-series are the general solutions around the singularities. Moreover, it has been proved that the Psi-series are convergent [16]. Therefore, it follows from the main theorem that the solutions of the Lorenz system never exhibit finite-time blow-up in the variable x.

Let us also note that the dominant truncation of the vector field $\hat{f}$ has two first integrals $\hat{J}_1 = y^2 + z^2$ and $\hat{J}_2 = x^2 - 2\sigma z$. From Proposition 4 and the definiteness of $\hat{J}_1$, it follows that the variables y, z never blow up.

5.3. *Fluid Dynamics Example*

In order to model the interaction between vorticity and shear in turbulent flow [26], Vieillefosse introduced a five dimensional ODE system whose blow-up shows that the flow of an incompressible and inviscid fluid diverges in a finite time. The existence of finite time blow-up is proven by decoupling the system and reducing the dynamics of one of its variables to a Hamiltonian dynamic with a simple potential. We show here how this result can be obtained in a straightforward way. The system reads (in our notation):

$$\dot{x}_1 = -(x_3 + x_4) \tag{37.a}$$

$$\dot{x}_2 = x_4 \tag{37.b}$$

$$\dot{x}_3 = -\tfrac{3}{2}x_5 + \tfrac{1}{2}x_1x_2 - \tfrac{1}{4}x_1^2 \tag{37.c}$$

$$\dot{x}_4 = \tfrac{1}{2}x_5 + \tfrac{1}{6}x_1x_2 - \tfrac{1}{3}x_2^2 \tag{37.d}$$

$$\dot{x}_5 = \tfrac{1}{3}x_4x_1 - \tfrac{2}{3}x_2x_4 \tag{37.e}$$

We find that there is a balance (α, p) characterized by the leading exponents p with leading order coefficients α:

$$p = (-2, -2, -3, -3, -4) \tag{38.a}$$

$$\alpha = (144, 72, -432, 144, 864) \tag{38.b}$$

Since the leading order coefficients are real, the general solution of this system will exhibit finite-time blow-up if the balance we have chosen indeed corresponds to a general solution. Checking, we find that the resonances are $r = 2$, $r = 3$, $r = 4$ and $r = 6$. Therefore, there exists a general Psi-series solution based on this balance. As a consequence of the main theorem, the general solution will exhibit finite time blow-up for some open set of real initial conditions. Moreover, the blow-up occurs on the orthant $\{+, +, -, +, +\}$ (see Proposition 2).

6. CONCLUSIONS

We have found necessary and sufficient conditions for finite time singularities for a large class of ODEs. These conditions rely on the analysis of the local series solutions around the singularities and can be expressed as a reality condition on the leading behavior of the solutions near blow-up. Roughly speaking, finite time blow-up will occur if and only if the dominant terms in the local general series are real. In order to find which series correspond to the general solution (among the plethora of local solutions) we investigated the resonances and the corresponding arbitrary coefficients of the Psi-series. This allowed us to find a homeomorphism between an an open set of initial conditions and open set of arbitrary constants. Moreover, we were also able to determine the location in phase-space where blow-up occurs and explore the relationship between the absence of finite time singularities and first integrals. To illustrate these different results we analyzed different examples from different fields of applied mathematics.

The class of systems considered here was constrained by the requirement that the Psi-series exist. As we already stated, this encompasses a large class of systems. However, we believe that this limitation is merely technical. Indeed, the results do not rely on the specific form of the Psi-series but only on the fact that they describe general solutions. The reality condition applies only to the most dominant terms near blow-up. Therefore, we conjecture that our main theorem is actually valid for a much larger class of systems and that the conditions on the leading exponents and resonances (as being rational numbers) could be relaxed to the case where they are real numbers. If this conjecture holds, it could provide a universal way of detecting the existence of blow-up for systems of ordinary differential equations.

Another limitation of our results is the fact that we considered, for the sake of simplicity, only general solutions of ODEs rather than singular solutions. This point was important in establishing the existence of the homeomorphism between initial conditions and arbitrary coefficients. However, our results could probably be extended by considering the possibility of blow-up for singular solutions. Indeed, some systems may exhibit finite-time blow-up only for constrained sets of initial conditions rather than open sets. The solutions based on these sets are not in the set of general solutions, however their asymptotic behavior near blow-up can still be analyzed by studying the balances (α, p) corresponding to singular solutions (that is the Psi-series with less than $n-1$ arbitrary coefficients). Similar results on the blow-up of particular solutions could then be obtained.

An interesting consequence of our main theorem is that the blow-up of a system ultimately depends only on the dominant behavior, that is the balance (α, p). These balances are computed from the knowledge of the dominant part of the vector field (in the case where the vector field is homogeneous, the dominant part is, roughly speaking, given by the terms of maximal degree only). Therefore, we see that the blow-up is controlled only by these terms and not by lower order terms (such as the linear terms for instance). The effect of the lower-order terms could be to create regions in phase-space where the solutions are bounded but they never manage to prevent the solution to blow-up in the entire phase-space. This point is important because in many instances the dominant part of a given vector field assumes a simple form and can be exactly integrated (by quadratures or by showing explicitly the existence of a set of first integrals). In turn, these explicit solutions can be used to compute an estimate of the blow-up time as a function of the initial conditions. This estimate becomes better as one approaches the blow-up point.

In a companion paper, we further use our results and apply them to systems appearing in magneto-hydrodynamics contexts where blow-up has not been proved previously [27].

It is well-known that near fixed points the solutions of a given system of ODEs essentially behave according to the linear part, and most of the subsequent dynamical analysis rely on perturbation expansions around the linear solutions (the normal form theory à la Poincaré–Dulac is based on this basic idea). The theory developed in this paper shows that the most nonlinear part of the vector field determines the behavior of the solutions near its singularities. We believe that a thorough understanding of the dynamics of unbounded systems can only be achieved by merging the two approaches and we hope that the ideas presented in this paper may provide a first step in this direction.

7. APPENDIX

We now prove Lemmas 1 and 2:

LEMMA 1. *The series given in* (7) *is a general formal solution to the system* (5).

LEMMA 2. *Let* $x=\Psi(\alpha, p, t; c_2, ..., c_m)$ *be a solution of* $\dot{x}=f(x)$ *around the singularity* t_* *containing* $(m-1)$ *arbitrary coefficients. If* $\alpha \in \mathbb{R}^n$, *and* $c_i \in \mathbb{R}\ \forall i=2, ..., m$ *then,* $a_{jk} \in \mathbb{R}^n\ \forall (j, k)$

Proof of Lemmas 1 *and* 2. In order to prove these lemmas, we first look at the form of the recursion relations.

We begin by writing an expression for the ith component of the n-dimensional Eq. (8), i.e.

$$\dot{x}_i=f_i(x)=\hat{f}_i(x)+\check{f}_i(x) \tag{39}$$

We also rewrite the Psi-series (7) for a single component of x:

$$x_i = \tau^{p_i}\left(\sum_{j=0}^{\infty}\sum_{k=0}^{N_j} a_{j,k}^{(i)}\tau^{j/q}(\log\tau)^k\right) \tag{40}$$

where $\tau = (t_* - t)$ and $a_{0,0} = \alpha$. The $a_{j,k}$ coefficients are now constants.

In order to obtain a generic recursion relation, it is necessary to introduce a specific form for $\hat{f}_i$ and $\check{f}_i$. Thus, let

$$\hat{f}_i(x) = a x_1^{M_1} x_2^{M_2} \cdots x_n^{M_n} \tag{41.a}$$

$$\check{f}_i(x) = b x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} \tag{41.b}$$

Without loss of generality, we have made both $\hat{f}_i$ and $\check{f}_i$ consist of only one polynomial term, but it will hopefully be apparent to the reader that the general form of the recursion relations which result from this would be unchanged for an arbitrary number of terms. Since $\hat{f}_i(x)$ is the dominant balance term, it will by definition have a leading-order power of τ which matches that of $\dot{x}_i$ (i.e. it will go as τ^{p_i-1}), while the leading order term of $\check{f}_i(x)$ will be at least one order higher (so that it goes as $\tau^{p_i-1+j/q}$ or higher). Thus, Eq. (39) becomes

$$\begin{aligned}
-\tau^{p_i-1}&\left[\sum_{\substack{j=0\\k=0}}\left(\frac{j}{q}+p_i\right)a_{j,k}^{(i)}\tau^{j/q}(\log\tau)^k + \sum_{\substack{j=0\\k=0}} k a_{j,k}^{(i)}\tau^{j/q}(\log\tau)^{k-1}\right]\\
&= a\tau^{p_i-1}\left[\left(\sum_{\substack{j=0\\k=0}} a_{j,k}^{(1)}\tau^{j/q}(\log\tau)^k\right)^{M_1}\left(\sum_{\substack{j=0\\k=0}} a_{j,k}^{(2)}\tau^{j/q}(\log\tau)^k\right)^{M_2}\cdots\right.\\
&\quad\left.\times\left(\sum_{\substack{j=0\\k=0}} a_{j,k}^{(n)}\tau^{j/q}(\log\tau)^k\right)^{M_n}\right]\\
&\quad + b\tau^{p_i-1+1/q}\left[\left(\sum_{\substack{j=0\\k=0}} a_{j,k}^{(1)}\tau^{j/q}(\log\tau)^k\right)^{m_1}\left(\sum_{\substack{j=0\\k=0}} a_{j,k}^{(2)}\tau^{j/q}(\log\tau)^k\right)^{m_2}\cdots\right.\\
&\quad\left.\times\left(\sum_{\substack{j=0\\k=0}} a_{j,k}^{(n)}\tau^{j/q}(\log\tau)^k\right)^{m_n}\right]
\end{aligned} \tag{42.a}$$

If we now consider collecting terms of order $\tau^{J/q+p_i-1}$, it is clear that these will only involve coefficients whose first index is less than J (i.e. $a_{j,k}$ with

$j<J$) except in the case where there is a product of a single $a_{J,k}$ times many $a_{0,0}$. Furthermore, this last case will only occur for the dominant balance term, as the highest j value for an $a_{j,k}$ appearing at this order in the $\check{f}_i$ term would be $j=J-1$. We therefore deduce that the recursion relation for the ith component of (8) at order $\tau^{J/q+p_i-1}(\log\tau)^K$ (valid only for $J\geqslant 1$) is:

$$\begin{aligned}
-\left(\frac{J}{q}+p_i\right) a^{(i)}_{J,K}-(K+1)\, a^{(i)}_{J,K+1} \\
&= M_1 a^{(1)}_{J,K}(a^{(1)}_{0,0})^{M_1-1}\,(a^{(2)}_{0,0})^{M_2}\,(a^{(3)}_{0,0})^{M_3}\cdots(a^{(n)}_{0,0})^{M_n} \\
&\quad + M_2 a^{(2)}_{J,K}(a^{(1)}_{0,0})^{M_1}\,(a^{(2)}_{0,0})^{M_2-1}\,(a^{(3)}_{0,0})^{M_3}\cdots(a^{(n)}_{0,0})^{M_n} \\
&\quad + \cdots + M_n a^{(n)}_{J,K}(a^{(1)}_{0,0})^{M_1}\,(a^{(2)}_{0,0})^{M_2}\,(a^{(3)}_{0,0})^{M_3}\cdots(a^{(n)}_{0,0})^{M_n-1} \\
&\quad + b^{(i)}_{J,K}(a_{j,k};\, j<J) \qquad (43.a)
\end{aligned}$$

which simplifies to

$$\begin{aligned}
-\left(\frac{J}{q}+p_i\right) a^{(i)}_{J,K} &= (K+1)\, a^{(i)}_{J,K+1}+\sum_{m=1}^{n}\left(\frac{\partial\hat{f}_i(a_{0,0})}{\partial x_m}\, a^{(m)}_{J,K}\right) \\
&\quad + b^{(i)}_{J,K}(a_{j,k};\, j<J) \qquad (44)
\end{aligned}$$

The $b^{(i)}_{J,K}$ term is some undetermined polynomial of various $a_{j,k}$ coefficients for which $j<J$ and $k\leqslant K$.

Equation (44) holds for any component i of the system, so upon dropping the index i and going back to the n-dimensional system, we get a matrix equation for the recursion relations:

$$\begin{aligned}
\left[-D\hat{f}(a_{0,0})-\frac{J}{q}I-\mathrm{diag}(p)\right] a_{J,K} \\
&= (K+1)\, a_{J,K+1}+b_{J,K}(a_{j,k};\, j<J,\, k\leqslant K) \qquad (45)
\end{aligned}$$

where $D\hat{f}(a_{0,0})$ is just the Jacobian matrix of $\hat{f}$ evaluated at $x=a_{0,0}$ (i.e. each x_i is evaluated at $a^{(i)}_{0,0}$). Defining the matrix $R\equiv[-D\hat{f}(a_{0,0})-\mathrm{diag}(p)]$ and dropping the capital indices, we shorten our notation to

$$\left[R-\frac{j}{q}I\right] a_{j,k}=(k+1)\, a_{j,k+1}+b_{j,k} \qquad (46)$$

This is just an n-dimensional linear system with constant coefficients. It always has a unique solution except when j/q is an eigenvalue of the matrix R. As discussed previously, these eigenvalues are the resonances, and we can see now why they correspond to arbitrary coefficients. One such resonance is always -1.

Proof of Lemma 1. We only need to show that (46) does in fact have a solution. Since we are assuming that our Psi-series expansion represents a general solution to the system, there must be $n-1$ non-negative rational resonances, not necessarily distinct (note that q is the l.c.d. of these resonances). For now, we will deal with the case where for every repeated eigenvalue, its algebraic multiplicity will equal its geometric multiplicity (i.e. the number of orthogonal eigenvectors associated with it will equal its multiplicity) so that the total number of arbitrary parameters contained in all the eigenvectors will still be $n-1$.

Let r_m/q denote the mth non-negative resonance. As long as $j<r_1$, $a_{j,k}$ will be zero except when $k=0$. Let us write the recursion relations at $k=0$ in the following form:

$$R_j a_{j,0} = a_{j,1} + b_{j,0} \tag{47}$$

where here and throughout the remainder of the appendix, we define $R_j \equiv R-(j/q)\, I$. When $0<j<r_1$, the matrix R_j is invertible, so a solution exists for any $a_{j,1}$ and all $b_{j,0}$. Recall that the $a_{0,0}$ coefficients are determined by balancing the leading order terms and, by assumption, there is no need for a logarithm at leading order. Therefore $b_{1,k}=0$ for all $k \geqslant 1$. But then the recursion relations at $j=1$ and $k \geqslant 1$ will be

$$R_1 a_{1,k} = (k+1)\, a_{1,k+1}, \tag{48}$$

so that, in order to avoid an infinite chain of linear equations, we must set $a_{1,k}=0$ for all $k \geqslant 1$. But then $b_{2,k}=0$ for all $k \geqslant 1$, so by the same argument $a_{2,k}=0$ for all $k \geqslant 1$. This process continues as we increase j so long as $j<r_1$, i.e. so long as R_j is invertible, and thus $a_{j,k}=0$ for all $k \geqslant 1$ whenever $j<r_1$. The introduction of the first logarithm may occur at $j=r_1$ if the recursion relation for $a_{r_1,0}$ fails to satisfy the solvability condition for the non-invertible matrix R_{r_1}. (In the case that $r_1=0$, there would be no logarithm until r_2. Having $r_1=0$ simply means that some of the components of $a_{0,0}$ are arbitrary. For this discussion, we will assume $r_1>0$.) It is possible that $b_{r_1,0}$ will not lie in the range of the matrix R_{r_1}. Let y_{r_1} denote a normalized null vector of R_{r_1} (assume for now that there is only one). Since we are currently operating under the assumption that the matrix R has a full set of eigenvectors, there will be no generalized eigenvectors at all, and thus the null vector(s) of R_{r_m} will never lie in the range of R_{r_m}. We can therefore solve

$$R_{r_1} a_{r_1,0} = a_{r_1,1} + b_{r_1,0} \tag{49}$$

even in the case where $b_{r_1,0}$ is *not* in the range of R_{r_1} by looking at the recursion relation at $k=1$:

$$R_{r_1} a_{r_1,1} = 0 \tag{50}$$

while still setting $a_{r_1,k} = 0$ whenever $k \geqslant 2$. The solution of (50) is $a_{r_1,1} = c y_{r_1}$, where c is an arbitrary constant. The solvability condition for (49) will now be

$$c\langle v \mid y_{r_1}\rangle + \langle v \mid b_{r_1,0}\rangle = 0 \tag{51}$$

where v is the null vector of the adjoint of R_{r_1} and $<|>$ denotes the inner product. This condition is automatically satisfied by choosing the heretofore arbitrary c to be

$$c = -\frac{\langle v \mid b_{r_1,0}\rangle}{\langle v \mid y_{r_1}\rangle}. \tag{52}$$

Since y_{r_1} does not lie in the range of R_{r_1}, the denominator of this expression will be non-zero. Thus, we have regained solvability by introducing one power of $\log \tau$ at this order. Note that we still a retain an arbitrary parameter, as the solution to (49) will be a particular solution plus $\tilde{c} y_{r_1}$ where $\tilde{c}$ is arbitrary. The reader can check that the same procedure works in the case where R_{r_1} has multiple orthogonal null vectors, with the number of arbitrary parameters retained equal to the number of null vectors.

For $j > r_1$, the $\log \tau$ introduced at $j = r_1$ may be raised to various powers for higher j values due to the non-linearities (of degree up to M) in the $b_{j,k}$ terms. Thus, when $r_1 < j < r_2$, higher and higher powers of $\log \tau$ may build up so that $b_{j,k} \neq 0$ for higher and higher values of k. However, as long as R_j is invertible, solutions for the coefficients will still exist. Let

$$\tilde{n}_{j,k} \equiv \sup_{n \in \mathbb{Z}_+} \{n \mid r_k \leqslant n r_j\}. \tag{53}$$

Then for $j = r_2$, the highest possible power of $\log \tau$ in the recursion relations will be $(\log \tau)^{\tilde{n}_{2,1}}$, i.e. the highest value of k for which $b_{r_2,k} \neq 0$ will be $\tilde{n}_{2,1} \equiv \tilde{n}$. As before, we propose only one power of $\log \tau$ beyond this, so that we get the following system of recursion relations:

$$R_{r_2} a_{r_2,\tilde{n}+1} = 0 \tag{54.a}$$

$$R_{r_2} a_{r_2,\tilde{n}} = (\tilde{n}+1)\, a_{r_2,\tilde{n}+1} + b_{r_2,\tilde{n}} \tag{54.b}$$

$$R_{r_2} a_{r_2,\tilde{n}-1} = (\tilde{n})\, a_{r_2,\tilde{n}} + b_{r_2,\tilde{n}-1} \tag{54.c}$$

$$\vdots$$

$$R_{r_2} a_{r_2,1} = 2a_{r_2,2} + b_{r_2,1} \tag{54.d}$$

$$R_{r_2} a_{r_2,0} = a_{r_2,1} + b_{r_2,0} \tag{54.e}$$

Let y_{r_2} be the only null vector of R_{r_2} and v the null vector of the adjoint $R^*_{r_2}$. Then the solution of (54.a) is $a_{r_2,\tilde{n}+1} = c_1 y_{r_2}$ with c_1 arbitrary. This can be used to satisfy the solvability condition of (54.b) by choosing

$$c_1 = -\frac{\langle v \mid b_{r_2,\tilde{n}} \rangle}{(\tilde{n}+1)\langle v \mid y_{r_2} \rangle}. \tag{55}$$

The solution of (54.b) will then be $a_{r_2,\tilde{n}} = c_2 y_{r_2} + x_p$, where x_p is the particular solution and c_2 is arbitrary. This can then be used to satisfy the solvability condition for (54.c) by demanding

$$c_2 = \frac{-(\langle v \mid b_{r_2,\tilde{n}-1} \rangle + \tilde{n}\langle v \mid x_p \rangle)}{(\tilde{n})\langle v \mid y_{r_2} \rangle} \tag{56}$$

and so on, so that all of the solvability conditions are met and we still have one arbitrary constant from $a_{r_2,0}$.

In general, we can predict the highest power of $\log \tau$ appearing in the recursion relations at $j = r_m$ (i.e. the highest k for which $b_{r_m,k} \neq 0$) by looking at the values of $\tilde{n}_{r_m,i}$ for $i < r_m$. Then, if the solvability conditions are not satisfied at this power of $\log \tau$, we must add at most one more power of $\log \tau$, assuming there are a complete set of eigenvectors. Thus, at $j = r_3$, the highest possible logarithmic power will be given by $(\log \tau)^{N_{r_3}}$, where

$$N_{r_3} = \sup\{\tilde{n}_{3,1} + 1, (\tilde{n}_{2,1} + 1)\, \tilde{n}_{3,2} + 1\}, \tag{57}$$

and so on. For brevity, if we let $\tilde{N} \equiv \sup_{j,k}\{\tilde{n}_{j,k}\}$, then a strict upper bound for the power of $\log \tau$ at order $j = r_m$ is given by B_m, where

$$B_m = \tilde{N}^{m-1} + \tilde{N}^{m-2} + \cdots + \tilde{N} + 1 \tag{58}$$

At any value of j in the recursion relations, the highest power of $\log \tau$ will always be bounded by some finite integer N_j, but $N_j \to \infty$ as $j \to \infty$.

Let us now consider the case in which one of the positive eigenvalues r_m/q of the matrix R has a multiplicity exceeding the number of eigenvalues that go with it. Assume there is only one eigenvector. Then the algebraic multiplicity of r_m/q is equal to the number of generalized eigenvectors (including the actual eigenvector) associated with the eigenvalue r_m/q. (In general, a chain of generalized eigenvectors can begin with every eigenvector if there is excess eigenvalue multiplicity.) Without loss of generality, let us assume an eigenvalue multiplicity of 3 with only one eigenvector. Using the variable y_{r_m} to denote the eigenvector (which is also the null vector of R_{r_m}), we have the following relationships for the generalized eigenvectors:

$$R_{r_m} y_{r_m} = 0 \tag{59.a}$$

$$R_{r_m} z_0 = y_{r_m} \tag{59.b}$$

$$R_{r_m} z_1 = z_0 \tag{59.c}$$

The vectors z_0 and z_1 are the generalized eigenvectors associated with y_{r_m}. All three of these vectors can be chosen orthonormal, and we will assume that this has been done here. It is a fact that, for the eigenvalue multiplicity of 3, there can be no more generalized eigenvalues than these. Thus, z_1 is *not* in the range of the matrix R_{r_m}, but both y_{r_m} and z_0 are. Once again, we let v be the null vector of $R^*_{r_m}$, and we will let κ be the highest value of k for which $b_{j,k} \neq 0$. Then the recursion relation at $k = \kappa$ is:

$$R_{r_m} a_{r_m,\kappa} = (\kappa + 1)\, a_{r_m,\kappa+1} + b_{r_m,\kappa} \tag{60}$$

Since y_{r_m} is now in the range of R_{r_m}, we cannot guarantee that this equation is solvable merely by letting $a_{r_m,\kappa+1} = c y_{r_m}$ with c arbitrary. If we were to attempt this, then letting v be the nullvector of $R^*_{r_m}$ as before, the Fredholm condition on the right hand side of (60) would be

$$\langle v \,|\, (\kappa+1)\, c y_{r_m} \rangle + \langle v \,|\, b_{r_m,\kappa} \rangle = \langle v \,|\, b_{r_m,\kappa} \rangle \equiv 0 \tag{61}$$

which may not be satisfied by $b_{r_m,\kappa}$. Thus, instead of adding one power of $\log \tau$ at $k = \kappa + 1$, we add three powers, giving the following additional recursion relations:

$$R_{r_m} a_{r_m,\kappa+3} = 0 \tag{62.a}$$

$$R_{r_m} a_{r_m,\kappa+2} = (\kappa + 3)\, a_{r_m,\kappa+3} \tag{62.b}$$

$$R_{r_m} a_{r_m,\kappa+1} = (\kappa + 2)\, a_{r_m,\kappa+2}. \tag{62.c}$$

The solutions to these are clearly

$$a_{r_m,\kappa+3} = c_0 y_{r_m} \tag{63.a}$$

$$a_{r_m,\kappa+2} = c_1 y_{r_m} + c_0(\kappa+3)\, z_0 \tag{63.b}$$

$$a_{r_m,\kappa+1} = c_2 y_{r_m} + c_1(\kappa+2)\, z_0 + c_0(\kappa+2)(\kappa+3)\, z_1 \tag{63.c}$$

where c_0, c_1, and c_2 are all arbitrary. Substituting (63.c) into (60) gives

$$R_{r_m} a_{r_m,\kappa} = (\kappa+1)\, c_2\, y_{r_m} + c_1(\kappa+1)(\kappa+2)\, z_0 + c_0(\kappa+1)(\kappa+2)(\kappa+3)\, z_1 + b_{r_m,\kappa}. \tag{64}$$

We know from our previous discussion of the generalized eigenvectors that both $\langle v \mid y_{r_m} \rangle = 0$ and $\langle v \mid z_0 \rangle = 0$, but $\langle v \mid z_1 \rangle \neq 0$. Thus, applying the Fredholm solvability condition to the right hand side of (64) gives

$$c_0(\kappa+1)(\kappa+2)(\kappa+3)\langle v \mid z_1 \rangle + \langle v \mid b_{r_m,\kappa} \rangle \equiv 0, \tag{65}$$

so that by choosing

$$c_0 = -\frac{\langle v \mid b_{r_m,\kappa} \rangle}{(\kappa+1)(\kappa+2)(\kappa+3)\langle v \mid z_1 \rangle} \tag{66}$$

the recursion relation (64) becomes solvable. Note that now the solution of (64) becomes

$$a_{r_m,\kappa} = c_3\, y_{r_m} + c_2(\kappa+1)\, z_0 + c_1(\kappa+1)(\kappa+2)\, z_1 \tag{67}$$

so that a proper choice of c_1 will satisfy the solvability condition of the recursion relation for $a_{r_m,\kappa-1}$, and so on. Three arbitrary parameters will be kept by the solution for $a_{r_m,0}$, so that we still get a full set of arbitrary parameters equal to the multiplicity of the eigenvalue r_m/q.

In general, if we let β equals the number of generalized eigenvectors (including the actual eigenvector) in the largest "chain" of generalized eigenvectors, then the maximum power of $\log\tau$ at $j = r_m$ will be given by $(\log\tau)^{\kappa+\beta}$ with κ as defined above.

Thus we see that for the class of systems we are dealing with (those that have rational leading order exponents and rational eigenvalues for the R matrix), a solution to the recursion relations derived from the Psi-series expansion (7) always exists with the proper number of arbitrary parameters, simply by adding a limited number of powers of $\log\tau$ at each resonance. This proves Lemma 1. ∎

Proof of Lemma 2. We are given that the leading order coefficients α, which we now call $a_{0,0}$, are real. The recursion relation (46) is a simple linear system, and for the class of systems we are considering, q and $\mathrm{diag}(P)$ are real. Therefore, the matrix R appearing in (46) is real. It is a fact that solutions to linear systems with real coefficients and real inhomogeneities are themselves real, so

$$a_{0,0} \in \mathbb{R}^n \Rightarrow b_{1,k} \in \mathbb{R}^n \quad \forall k \Rightarrow a_{1,k} \in \mathbb{R}^n \quad \forall k \tag{68}$$

as long as $1 < r_1$, where it is to be understood throughout this proof that the only allowable values for k and j are non-negative integers. (Recall that, by assumption, $a_{0,k} = 0$ for all $k \geqslant 1$.) If $r_1 = 1$, the solutions $a_{1,k}$ still consist of real values but with arbitrary coefficients which, *by hypothesis*, we choose to be real. In general,

$$a_{j,k} \in \mathbb{R}^n \quad \forall k \Rightarrow b_{j,k} \in \mathbb{R}^n \quad \forall k \Rightarrow a_{j+1,k} \in \mathbb{R}^n \quad \forall k \tag{69}$$

as long as $(j+1)/q$ is not a resonance. Thus we can say that, for all $j < r_1$ and all non-negative integers k, $a_{j,k} \in \mathbb{R}^n$. This implies that $b_{r_1,k} \in \mathbb{R}^n$ for all k, so that the singular linear system for $a_{r_1,k}$ consists of only real valued coefficients. This means that the solutions $a_{r_1,k}$ will be real but with arbitrary coefficients. Since, by hypothesis, we choose all arbitrary coefficients to be real, $a_{r_1,k}$ will be real for all k. This will be true at any resonance r_m/q for which $a_{j,k}$ is real for all $j < r_m$ and for all k. Thus we have that

$$a_{j,k} \in \mathbb{R}^n \quad \forall k \Rightarrow b_{j,k} \in \mathbb{R}^n \quad \forall k \Rightarrow a_{j+1,k} \in \mathbb{R}^n \quad \forall k \tag{70}$$

even when $(j+1)/q$ is a resonance, since, by hypothesis, we always choose the arbitrary coefficients to be real. Therefore

$$a_{0,0} \in \mathbb{R}^n \Rightarrow a_{j,k} \in \mathbb{R}^n \quad \forall j, \quad \forall k. \tag{71}$$

This proves Lemma 2. ∎

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