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KOVALEVSKAYA RODS AND KOVALEVSKAYA WAVES

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The Kirchhoff analogy for elastic rods establishes the equivalence between the solutions of the classical spinning top and the stationary solutions of the Kirchhoff model for thin elastic rods with circular cross-sections. In this paper the Kirchhoff analogy is further generalized to show that the classical Kovalevskaya solution for the rigid body problem is formally equivalent to the solution of the Kirchhoff model for thin elastic rod with anisotropic cross-sections (elastic strips). These Kovalevskaya rods are completely integrable and are part of a family of integrable traveling waves solutions for the rod (Kovalevskaya waves). The analysis of homoclinic twistless Kovalevskaya rod reveals the existence of a three parameter family of solutions corresponding to the Steklov and Bobylev integrable case of the rigid body problem. Furthermore, the existence of these integrable solutions is discussed in conjunction with recent results on the stability of strips.

Special Kovalevskaya Edition

1. Introduction

In 1889, Sophia Kovalevskaya made a remarkable contribution to problem of rigid body motion [1]. On the one hand she found a new integrable case for the Euler equations and gave a complete description of the solutions in terms of hyperelliptical integrals. On the other hand, she gave some evidences that no other integrability case exists other than the ones previously known. Despite the importance of her contribution in Classical Mechanics, the physical application of her case in terms of spinning tops or gyrostats remains doubtful. In a later paper she gave a description (first provided by A. Schwarz at the request of Weierstrass [2]) of a rigid body satisfying her hypotheses. However, it seems that the construction (two right circular cylinders with parallel axes) was rather artificial and to the best of our knowledge there has been no modern attempt to use her result in a real physical setting.

In this paper, we identify a physical system whose mathematical model satisfies Kovalevskaya's hypotheses. The system in question is not a rigid body but an elastic strip. The link between the two physical systems is provided through a modification of the so-called "Kirchhoff analogy" which identifies the *motion* of symmetric spinning tops with the *shape* of stationary thin inextensible elastic filaments with circular cross sections governed by linear elasticity. Within the Kirchhoff model of elastic rods we show that the shape of some elastic strips (rods with non circular cross sections) are given by Kovalevskaya's case of integrability. Rather than giving explicit solutions for the shape, we restrict our analysis to localized twistless solution (that is, solutions which asymptotically connect solutions with constant curvature and torsion). We explicitly compute these solutions and show that they exist not only at the Kovalevskaya point but on curves of points in parameter space. We identify these

Mathematics Subject Classification

curves with the Steklov and Bobylev case of partial integrability for the rigid body motion — a three parameter family of solutions. We show that these solutions are important for the understanding of the dynamics of elastic strips and are actually part of a 4-parameter family of traveling waves solutions for the dynamical Kirchhoff equations.

1.1. Euler equations

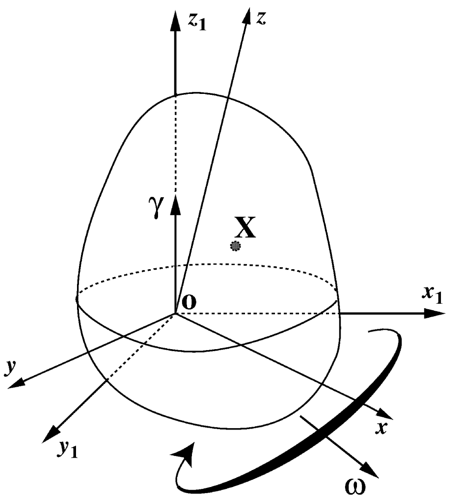

Fig. 1. A rigid body, the frame (x, y, z) is attached to the body whose center of mass is located at the point $\mathbf{X}$. The frame (x_1, y_1, z_1) is fixed in space. The vector $\boldsymbol{\gamma}$ gives the position of the unit vector along z_1 in the moving coordinates and $\boldsymbol{\kappa}$ is the angular velocity of the body (also in the moving frame).

Consider a rigid body in a gravity field moving around a fixed point $\mathbf{O}$. By a proper choice of units we set the product of the constant of gravity g and the mass of the body such that $mg = 1$. Consider now two cartesian frames, the first one *fixed* in space (x_1, y_1, z_1), the second one, (x, y, z), *moving* with the rigid body and directed along the principal axes of inertia of the body about the point $\mathbf{O}$, the principal moment of inertia are the eigenvalues of the inertia tensor and are denoted $\mathbf{I} = (I_1, I_2, I_3)$. Let $\boldsymbol{\gamma}$ be the coordinates of a unit vector along the z_1 axis and $\boldsymbol{\kappa}$ the vector of instantaneous rotation, both in the the moving basis. The center of mass of the body is located in the moving frame at the point $\mathbf{X} = (X_1, X_2, X_3)$. Since $\mathbf{I}$ and $\mathbf{X}$ are fixed, the motion is determined by the two vectors $\boldsymbol{\kappa}$ and $\boldsymbol{\gamma}$. (see Fig. 1). It is a standard matter (See for instance [3]) to show that the *Euler–Poisson equations* describing the motion of the rigid body about the fixed point are:

$$\mathbf{I}\dot{\boldsymbol{\kappa}} + \boldsymbol{\kappa} \times (\mathbf{I}\boldsymbol{\kappa}) = \mathbf{X} \times \boldsymbol{\gamma}\,, \tag{1.1 a}$$

$$\dot{\boldsymbol{\gamma}} + \boldsymbol{\kappa} \times \boldsymbol{\gamma} = 0\,. \tag{1.1 b}$$

The Euler–Poisson equations admit for all values of the 6 parameters $I_1, I_2, I_3, X_1, X_2, X_3$, three first integrals:

$$C_1 = \boldsymbol{\gamma}.\boldsymbol{\gamma} = 1\,, \tag{1.2 a}$$

$$C_2 = \frac{1}{2}(\mathbf{I}\boldsymbol{\kappa}).\boldsymbol{\kappa} + \mathbf{X}.\boldsymbol{\gamma}\,, \tag{1.2 b}$$

$$C_3 = (\mathbf{I}\boldsymbol{\kappa}).\boldsymbol{\gamma}\,. \tag{1.2 c}$$

The first relation expresses the fact that $\boldsymbol{\gamma}$ is a unit vector, the second constant is a form of the energy conservation and the third one is the conservation of the vertical component of the angular momentum (the force being vertical).

Prior to Kovalevskaya's work, this problem was only solved in the three following cases:

1. The complete symmetric case: $I_1 = I_2 = I_3$ with fourth integral:

$$C_4 = \boldsymbol{\kappa}.\mathbf{X}\,. \tag{1.3}$$

2. The Euler–Poinsot case: $X_1 = X_2 = X_3 = 0$, that is the center of gravity is the fixed point:

$$C_4 = (\mathbf{I}\boldsymbol{\kappa}).(\mathbf{I}\boldsymbol{\kappa})\,. \tag{1.4}$$

3. The Lagrange–Poisson case: $X_1 = X_2 = 0$ and $I_1 = I_2$, a symmetric top with the center of gravity along the z-axis:

$$C_4 = \kappa_3\,. \tag{1.5}$$

In 1889, Kovalevskaya showed that the rigid body motion was integrable in the case: $I_1 = I_2 = 2I_3$ and $X_3 = X_2 = 0$ with fourth integral[1, 4]:

$$C_4 = \left\{(\kappa_1 + i\kappa_2)^2 + X_1(\gamma_1 + i\gamma_2)\right\}\left\{(\kappa_1 - i\kappa_2)^2 + X_1(\gamma_1 - i\gamma_2)\right\} . \tag{1.6}$$

Her work was so remarkable that it won her the Bordin prize (1888).

2. The Kirchhoff model for elastic strips

In order to show the equivalence between the Kovalevskaya solutions for the rigid body mechanics and the solutions of the Kirchhoff model for stationary strips, we now present the kinematics and dynamics of elastic rods.

2.1. Space Curves and Ribbons

We consider a dynamical *space curve* $\mathbf{R}(s,t)$ parameterized by its arc length s and time t. A *strip* is defined as a space curve $\mathbf{R}(s,t)$ together with a smooth unit vector field $\mathbf{d}_2(s,t)$ orthogonal to that curve. The unit tangent vector, $\mathbf{d}_3 \equiv \mathbf{t}$, together with the field $\mathbf{d}_2(s,t)$ and $\mathbf{d}_1 = \mathbf{d}_2 \times \mathbf{d}_3$, forms a right-handed orthonormal basis. The components of the derivatives of this basis $\mathbf{d}_1$, $\mathbf{d}_2$ and $\mathbf{d}_3$ with respect to arc length s and time t expressed in the local basis form, respectively, the *twist vector* $\mathbf{k}(s,t) = k_1\mathbf{d}_1 + k_2\mathbf{d}_2 + k_3\mathbf{d}_3$ and the *spin vector* $\mathbf{w}(s,t) = w_1\mathbf{d}_1 + w_2\mathbf{d}_2 + w_3\mathbf{d}_3$:

$$\mathbf{d}_i' = \mathbf{k} \times \mathbf{d}_i\,, \qquad i = 1,2,3\,, \tag{2.1 a}$$

$$\dot{\mathbf{d}}_i = \mathbf{w} \times \mathbf{d}_i\,, \qquad i = 1,2,3\,, \tag{2.1 b}$$

with prime denoting differentiation with respect to arc length and dot denoting differentiation with respect to time. The component k_3 is the *twist density* and defines the amount of rotation of the local basis $(\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3)$ around the tangent vector $\mathbf{d}_3$ as s increases. The twist vector components can be expressed as functions of the angle φ between the vector $\mathbf{d}_1$ and the normal to the curve, the Frenet curvature k, and the torsion τ:

$$(k_1, k_2, k_3) = (k\sin\varphi, k\cos\varphi, \tau + \partial_s\varphi)\,. \tag{2.2}$$

2.2. The Kirchhoff Equations

The required assumptions are: (i) The filament is *thin*, *i. e.* the width of any cross-section is much smaller than other length scales (e. g. $1/|\mathbf{k}|$); (ii) The rod is unshearable and inextensible, *i. e.* each cross section remains normal to the axial space curve and can be identified by its arc-length coordinate; (iii) The elastic stresses are linear in the strains. Then, by standard arguments one can derive the evolution of the force, $\mathbf{g}(s,t)$ and the moment $\mathbf{m}(s,t)$ acting on each cross-section [5, 6]:

$$\mathbf{g}' = \rho\mathscr{A}\,\ddot{\mathbf{R}}\,, \tag{2.3 a}$$

$$\mathbf{m}' + \mathbf{d}_3 \times \mathbf{g} = \rho\left(I_2\mathbf{d}_1 \times \ddot{\mathbf{d}}_1 + I_1\mathbf{d}_2 \times \ddot{\mathbf{d}}_2\right)\,, \tag{2.3 b}$$

where ρ is the (constant) mass per unit volume of the rod and $\mathscr{A}$ the cross sectional area; the quantities I_1 and I_2 are the principal moments of inertia of the cross-section. These equations are closed by using the constitutive relation of linear elasticity:

$$\mathbf{m} = EI_1k_1\mathbf{d}_1 + EI_2k_2\mathbf{d}_2 + \mu Jk_3\mathbf{d}_3\,. \tag{2.4}$$

The products EI_1 and EI_2 are called the *principal bending stiffnesses* of the rod, and μJ is the *torsional stiffness*. The elastic material is determined by its Young's modulus E and shear modulus μ;

the parameter J depends on the cross-section shape. In the particular case of a circular cross-section, one has:

$$I_1 = I_2 = \frac{J}{2} = \frac{\pi R^4}{4}\,, \tag{2.5}$$

where R is the radius of the cross-section.

2.3. Scaling

We can further simplify the model by introducing the length, time, and mass scales:

$$[L] = \sqrt{\frac{I_1}{\mathscr{A}}}\,, \qquad [T] = \sqrt{\frac{\rho I_1}{E\mathscr{A}}}\,, \qquad [m] = \rho\sqrt{\mathscr{A} I_1}\,, \tag{2.6}$$

and making the replacements

$$\frac{\partial}{\partial s} \to \sqrt{\frac{\mathscr{A}}{I_1}}\frac{\partial}{\partial s}\,, \qquad \mathbf{k} \to \sqrt{\frac{\mathscr{A}}{I_1}}\mathbf{k}\,, \qquad \frac{\partial^2}{\partial t^2} \to \frac{E\mathscr{A}}{\rho I_1}\frac{\partial^2}{\partial t^2}\,, \tag{2.7 a}$$

$$\mathbf{g} \to E\mathscr{A}\ \mathbf{F}\,, \qquad \mathbf{m} \to E\sqrt{\mathscr{A} I_1}\mathbf{m}\,. \tag{2.7 b}$$

The Kirchhoff equations reduce to the dimensionless form

$$\mathbf{g}' = \ddot{\mathbf{R}}\,, \tag{2.8 a}$$

$$\mathbf{m}' + \mathbf{d}_3 \times \mathbf{g} = a\ \mathbf{d}_1 \times \ddot{\mathbf{d}}_1 + \mathbf{d}_2 \times \ddot{\mathbf{d}}_2\,, \tag{2.8 b}$$

$$\mathbf{m} = k_1\mathbf{d}_1 + ak_2\mathbf{d}_2 + bk_3\mathbf{d}_3\,, \tag{2.8 c}$$

where the constant $a = \frac{I_1}{I_2}$ measures the asymmetry of the cross-section. Our convention is to orient the vector fields $\mathbf{d}_1$ and $\mathbf{d}_2$ such that I_1 and I_2 are, respectively, the larger and smaller bending stiffnesses. In this case, we have:

$$0 < a \leqslant 1\,, \tag{2.9}$$

the value 1 being reached in the dynamically symmetric case where the moments of inertia are identical, the scaled radius of circular cross-sections is then equal to 2. Within the framework of linear elasticity theory [7, 8, 9] it is possible to compute a and b for a given cross-section shape. For instance, if we consider elliptic cross-sections with semi-axes A and B ($A < B$), we have:

$$a = \frac{A^2}{B^2}\,, \qquad b = \frac{1}{1+\sigma}\frac{2a}{1+a}\,. \tag{2.10}$$

The scaled semi-axes are, respectively, $2\sqrt{a}$ and 2. Other values of a and b for various shapes are represented on Fig. 2.

2.4. Negative Poisson Ratios

In order to obtain elastic filaments in the parameter space region $b > a$, one is forced, in general, to consider elastic material with negative Poisson ratios. At first sight, this might seem unphysical, however in the last decade many new materials have been found to have negative effective Poisson ratios. For instance, experimental measurements of the ratio between bending and twist coefficients for biological molecules such as DNA strands) are such that $0.7 < \frac{EI}{\mu J} < 1.5$, (that is, within our formalism, $a = 1$ and $0.7 < b < 1.5$) [10]. It has also been shown that rods with noncircular cross sections and high intrinsic twist may actually obey the Kirchhoff equations with $b > 1$ [11]. In any case, we show in this paper that the analysis of the integrable case of elastic rods, whether in the physical domain or not, sheds interesting lights on the dynamics of strips and rods in general.

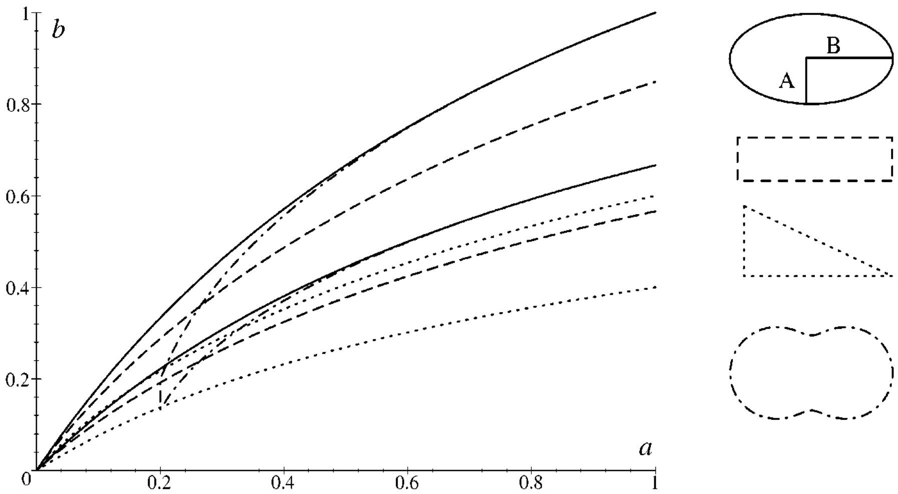

Fig. 2. The domains covered in the (a, b) plane by various cross-section shapes with $0 \leqslant \sigma \leqslant \frac{1}{2}$ are enclosed in black lines (solid = ellipses, dash = rectangles, dot = right triangles, dash-dot = limaçons).

3. The Kovalevskaya rod

3.1. The Kirchhoff analogy

In the case where the rod has a symmetric cross-section (for instance a homogeneous rod with circular cross-section), the principal moments of inertia I_1 and I_2 are equal (that is, $a = 1$) and it is well-known that the static Kirchhoff model is formally equivalent to the Euler–Poisson equation in the integrable Lagrange–Poisson case. This equivalence, known as *the Kirchhoff analogy*, is achieved by identifying the axes of the top with the tangent vector of the rod, the time with the rod's arc length, $\mathbf{g}$ with $\boldsymbol{\gamma}$ and $\boldsymbol{\kappa}$ with $\mathbf{k}$ (See Table 1). It is of interest to note that in the case of planar motion (where the rigid body reduces to a simple pendulum) this analogy was already known by James Bernoulli and Euler [12, art. 620]. By using the classical results on the spinning tops, it is therefore possible to complete the integration of the stationary solutions of the Kirchhoff equations in the case $a = 1$ and classify all solutions. Such a classification has been performed in [13] where each type of stationary rod is associated to a given type of orbit for the spinning top. For instance on Fig. 3, we show the spinning top orbit associated with a 3–5 torus knot.

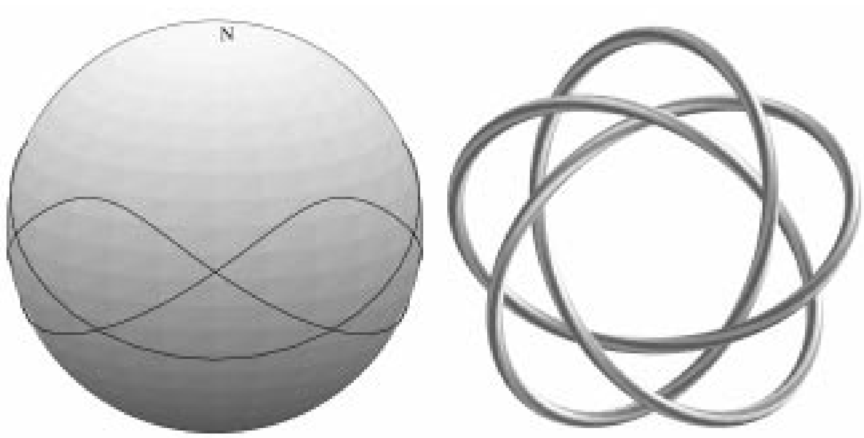

Fig. 3. A $5-3$ torus knot (right) together with the equivalent orbit obtained from the Kirchhoff analogy (left), the curve represents the curve drawn by the axis of the top on a sphere centered at the fixed point.

Symbol	Kirchhoff equations	Euler–Poisson equations
$\mathbf{d}_3$	Unit tangent vector	Unit vector joining the fixed point to the center of mass
$(\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3)$	Basis attached to the rod	Basis attached to the solid body
s	Arc length	Time
$\mathbf{g} \leftrightarrow \boldsymbol{\gamma}$	Tension	Force equal and opposite to gravity
$\mathbf{m}$	Moment	Angular momentum
$\boldsymbol{\kappa} \leftrightarrow \mathbf{k}$	Twist vector	Angular velocity vector
$EI_1 \leftrightarrow I_1$, $EI_2 \leftrightarrow I_2$	Principal bending stiffnesses	Principal moments of inertia in directions orthogonal to $\mathbf{d}_3$
$\mu J \leftrightarrow I_3$	Torsional stiffness	Principal moment of inertia along $\mathbf{d}_3$
a	Bending stiffnesses ratio	Ratio of the moments of inertia in directions orthogonal to $\mathbf{d}_3$
b	Scaled torsional stiffness	Scaled moment of inertia along $\mathbf{d}_3$

Table 1. The Kirchhoff analogy: between rigid bodies and static filaments.

3.2. Initial versus boundary conditions

There is, however, a major difference between spinning tops and stationary solutions of the Kirchhoff model. For the spinning top a solution is specified by a choice of *initial conditions*, that is, the position and velocity of the rigid body at a given time. In the case of elastic rod, the specification of the curvature and force vectors together with their derivatives at one point of the rod is not a realistic physical setup. Indeed, in general one is interested in finding the shape of the rod held at two distinct points in space. Moreover, in general, the rod will be of finite length (even though the analysis of infinite rods with conditions at infinity also provide useful information). Therefore, the problem in the theory of rods (or, for that matter, elastic materials in general) is to find the shape of the rods for given *boundary conditions*. These conditions could be on the vectors themselves and/or on their derivatives (for instance in the case of a clamped rod, one controls the derivatives of the curvature at one point). If moreover, one is interested in finding closed solutions (such as the one depicted in Fig. 3), then one has to look for periodic solutions of the curvature and force vectors and enforce a condition on the integrals of the variables (giving the actual position of the rod in space).

There is yet another major difference between the two problems. In the theory of elastic rods, some of the most interesting types of solutions are localized solutions. These solutions, homoclinic in the curvature vector, typically connect a straight rod to itself. They are known to play a crucial role in the understanding of the universal buckling instability where a rod under constraints changes shape, writhes and eventually coils on itself [14, 15, 16]. Therefore, in order to study these solutions one looks for homoclinic solutions of the stationary equations by specifying boundary conditions at infinity (one such solution for symmetric rod is shown on Fig. 4). According to the Kirchhoff analogy, the corresponding solutions for spinning tops, connect asymptotically the north pole to itself (see Fig. 4). These solutions are clearly unstable for the spinning top and unlikely to be observed as a typical gyroscopic motion; they are not important for the understanding of the motion; few researchers have actually focussed their investigation on them and they are at best mentioned in passing as an amusing, aberrant, limit of periodic solutions.

3.3. The Kirchhoff analogy for elastic strips

In order to obtain an analogy between the Kovalevskaya tops and elastic strips, we must slightly modify Kirchhoff's analogy. Indeed, if we use the analogy as specified on Table 1, we find that the Kovalevskaya case corresponds to ($a = 1,\ b = 2$), point at which we already know that the system is integrable ($a = 1$ is the Lagrange–Poisson case). The main important point in the Kovalevskaya case

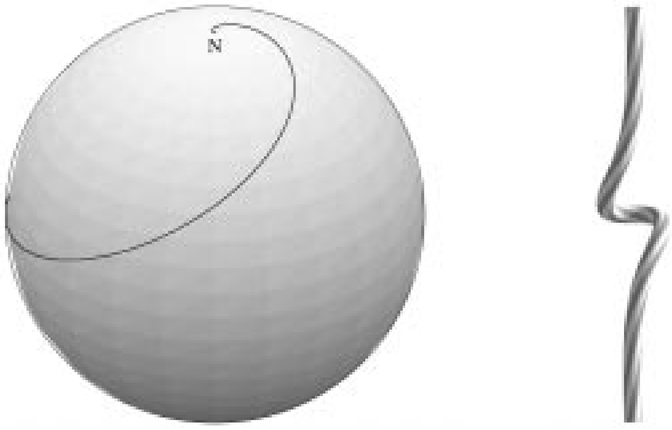

Fig. 4. A homoclinic solution for the spinning top and a localized elastic rod.

is that there exists an axis D, originating from the fixed point, along which the moment of inertia is half as large as the moment of inertia along directions perpendicular to D. In particular, it implies that we can choose, without loss of generality, two of the center of mass coordinates to vanish identically. In the Kirchhoff analogy we have $X_3 > 0$ and $X_1 = X_2 = 0$. The variation on the Kirchhoff analogy is as follows:

$$g_1 = \gamma_3\,, \qquad g_2 = -\gamma_1\,, \qquad g_3 = -\gamma_2\,, \tag{3.1 a}$$

$$k_1 = \kappa_3\,, \qquad k_2 = -\kappa_1\,, \qquad k_3 = -\kappa_2\,, \tag{3.1 b}$$

$$X_1 = 0\,, \qquad X_2 = 1\,, \qquad X_3 = 0\,, \tag{3.1 c}$$

$$I_1 = a\,, \qquad I_2 = b\,, \qquad I_3 = 1\,, \tag{3.1 d}$$

Now, the condition for the Kovalevskaya case reads for the Kirchhoff equations:

$$a = \frac{1}{2}\,, \qquad b = 1\,, \tag{3.2}$$

that is an elastic strip with aspect ratio $\frac{1}{2}$. Another choice of axis (swapping the role of X_1 and X_2) leads to the values $a = b = 2$, outside of the domain of definition of a $(0 < a \leqslant 1)$.

For $a = \frac{1}{2}$, $b = 1$, it is straightforward to show that the Kirchhoff system is algebraically integrable with four first integrals (using the transformation (3.1 a) and the known first integrals of the Kovalevskaya top):

$$C_1 = k_1^2 + \frac{1}{2}k_2^2 + k_3^2 + 2g_3\,, \tag{3.3 a}$$

$$C_2 = g_1^2 + g_2^2 + g_3^2\,, \tag{3.3 b}$$

$$C_3 = k_1 g_1 + \frac{1}{2}k_2 g_2 + k_3 g_3\,, \tag{3.3 c}$$

$$C_4 = (k_3^2 - k_1^2 - 2g_3)^2 + (2k_1k_3 - 2g_1)^2\,. \tag{3.3 d}$$

Since the trace of divergence of the vector field vanishes identically, the unity is a conserved density, Jacobi's last multiplier theorem can be used to build a fifth first integral. Eventually, if we follow the step of Kovalevskaya, the solution can be written in terms of hyperelliptical integrals. However, it is also well-known that this construction is extremely tedious and cumbersome and might not provide any meaningful information on the shape of rods. Rather than building the general solutions, we focus on a class of solutions which are known to be of prime importance for the understanding of rods dynamics.

4. Twistless Kovalevskaya rods

Guided by the existence of a completely integrable system in the case $a = \frac{1}{2},\ b = 1$, we further restrict our analysis of Kovalevskaya rods to twistless solutions. In this case k_3 represents only the torsion. It is not obvious that solution of this type with non-constant curvature and torsion actually exist and we look for conditions on the parameter a and b. As discussed earlier, we are particularly interested in localized twistless solutions. To do so, we consid er the static form of the system (2.8) and introduce the components of the force in the local basis $\mathbf{g} = g_1\mathbf{d}_1 + g_2\mathbf{d}_2 + g_3\mathbf{d}_3$: and projecting (2.8) along this basis we obtain a system of six equations involving the six unknowns $g_1,\ g_2,\ g_3,\ k_1,\ k_2$ and k_3:

$$g_1' + k_2 g_3 - k_3 g_2 = 0\,, \tag{4.1 a}$$

$$g_2' + k_3 g_1 - k_1 g_3 = 0\,, \tag{4.1 b}$$

$$g_3' + k_1 g_2 - k_2 g_1 = 0\,, \tag{4.1 c}$$

$$g_2 = k_1' + (b-a)k_3 k_2\,, \tag{4.1 d}$$

$$g_1 = a k_2' + (b-1)k_3 k_1\,, \tag{4.1 e}$$

$$b k_3' + (a-1)\ k_1 k_2 = 0\,. \tag{4.1 f}$$

In the case of a circular cross-section ($a = 1$) we immediately see from (4.1 f) that $\partial_s k_3 = 0$ and hence the twist density is constant along the rod. This property is crucial for the integrability of the system. Written in terms of the twist φ, the equation for k_3 reads:

$$b k_3' + (a-1)\ k\cos(\varphi)\sin(\varphi) = 0\,. \tag{4.2}$$

The torsion $k_3 = \tau$ becomes a first integral ($k_3 = C_1$) when $\varphi = j\frac{\pi}{2},\ j \in \mathbb{Z}$. First, consider the case where $\varphi = j\pi$. Then the equations for the forces $\mathbf{g}$ can be solved:

$$g_1 = -ak'\,, \tag{4.3 a}$$

$$g_2 = (b-a)k k_3\,, \tag{4.3 b}$$

$$g_3 = -\frac{k^2}{a} + aC_2\,. \tag{4.3 c}$$

Equation (4.1 e) then becomes

$$(b-2a)k_3 k' = 0\,. \tag{4.4}$$

Since we are interested in non planar ($\tau \neq 0$) non-constant ($k\tau$ non-constant) solutions, we must take $b = 2a$ and we obtain an equation for the curvature:

$$k'' = k(C_2 - C_1^2) - \frac{1}{2}k^3\,. \tag{4.5}$$

That is, the curvature evolves in a quartic, Duffing-like, potential. In particular it admits a homoclinic orbit:

$$k = 2\sqrt{C_2 - C1^2}\operatorname{sech}\left(\sqrt{C_2 - C1^2}s\right) \tag{4.6}$$

Hence we have found a solution with three arbitrary constants ($C_1,\ C_2$ and the energy of the Duffing oscillator). Once the curvature and torsion is known, one can reconstruct the curve by solving the twist equations (???) and integrating once the tangent vector (an exact form for the curve can also be obtained in terms of cylindrical coordinates as shown in [13]). An example of such a Kovalevskaya rod is shown on Fig. 5.

We see that the condition $b = 2a$ is actually more general than the condition for the Kovalevskaya case. This is due to the fact that we have restricted the solution set by looking at solutions with constant k_3. The corresponding case for spinning tops ($J_1 = 2J_2$ with $X_2 = X_3 = 0$) has actually been studied by V. A. Steklov and D. Bobylev (1896) who found a solution with three arbitrary constants (see [17, Sect. 8.4]).

The other choice of twist $\varphi = \frac{\pi}{2}$ or $\varphi = \frac{3\pi}{2}$ leads to the condition $b = 2$ with a curvature equation: $k'' = k(aC_2 - C_1^2) - \frac{1}{2}k^3$ and a similar homoclinic orbit (See Fig. 5).

The essential difference between the two types of solution is the orientation of the strip. In the case $b = 2a$, the central part of the localized solution is a binormal helix (a helical strip where the axis corresponding to the largest moment of inertia follows the binormal vector) modulated on long scales by a sech-like solution. In the case $b = 2$, the solution is a modulated normal helix (the axis corresponding to the largest moment of inertia now follows the normal vector). Interestingly enough, the two lines in parameter space meet at the point $a = 1, b = 2$ which is also a Kovalevskaya point (corresponding to the choice $X_1 = X_2 = 0$.)

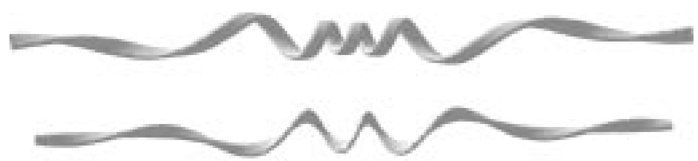

Fig. 5. The two localized exact solutions. The first one is a Kovalevskaya rod, that is a binormal homoclinic solution (top), the second one is a normal homoclinic solution (bottom).

4.1. More integrable strips

We have seen that integrable cases of the rigid body motion correspond to interesting shapes of elastic strips. A natural question is then: Are there any other cases of partial integrability of the rigid body motion leading to partially integrable strips? The condition for the equivalence between rods and rigid bodies is that two of the coordinates of the center of mass $\mathbf{X}$ vanish identically. Hence, we can look at all the known cases of partial integrability of the rigid body dynamics that satisfy this conditions. A rather complete (but probably not exhaustive) list of integrable cases was given by Leimanis (1965). We used his description together with the three possible choices of Kirchhoff analogies (depending on the choices of the component X_i that does not vanish) to find all possible candidates and found several new interesting integrable strips, namely:

1. The Goryachev – Chaplygin top: $a = \frac{1}{4}, b = 1$;
2. The second case of Goryachev: $b = 16a\dfrac{a-1}{8a-9}$;
3. The case of N. Kowalewski: $b = 18a\dfrac{a-1}{9a-10}$;
4. The second case of Chaplygin: $4a = 9(2a-b)(2-b)$;
5. The case of Corlis and Field: $b = 16a\dfrac{a-1}{8a-9}$.

These conditions on the parameters are represented on Fig. 6. The type of elastic rods that can be obtained from these integrable cases is not yet known and probably deserve further analysis.

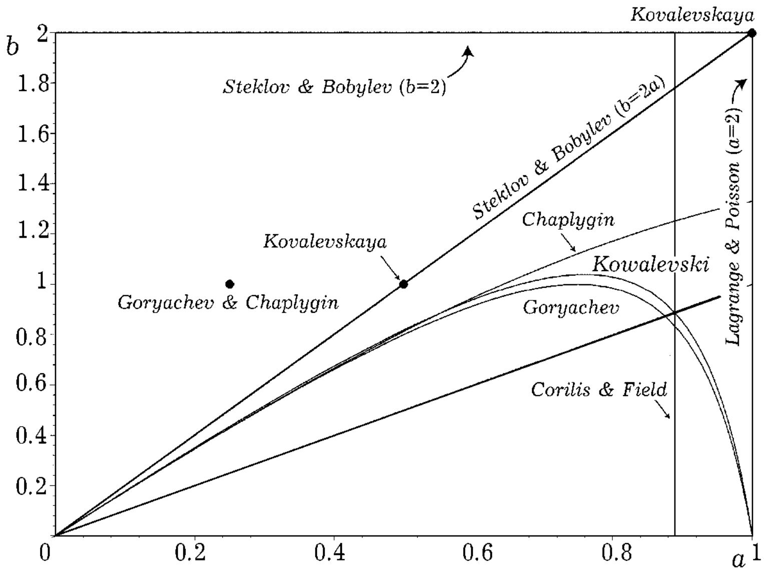

Fig. 6. The integrable cases of the Kirchhoff equation, the names correspond to the Classification of Leimanis (1965). The equations are completely integrable only in the case of Kovalevskaya and Lagrange–Poisson, the other cases are all cases of partial integrability.

4.2. Dynamic interpretation of the integrable case

The existence of integrable solutions is a nice mathematical property but as such might not be physically relevant. In this particular case, these solutions shed new lights on the dynamics. Indeed, in [18], the problem of the dynamical buckling instability of twisted elastic strips under tension has been considered. It has been shown that there exist two different types of bifurcation (Hamiltonian-Hopf or Hamiltonian-pitchfork) depending on the ratio $\frac{b}{a}$. A nonlinear analysis for the bifurcation modes was developed and an amplitude equation was derived. In the simplest case, these amplitude equations have the form of a complex Duffing oscillator for the amplitude of the unstable mode. It was found that the global behavior of elastic strips close to the bifurcation depends on the sign of $b-2a$, more precisely, the quantity $b-2a$ controls the localization or delocalization of unstable modes that can be observed in rods. This is precisely where the homoclinic solutions exists. Moreover, it was noticed in [19], that the point $a=1, b=2$ is a bifurcation point of codimension 3 and that there exist after the bifurcation two types of localized solutions, the normal and binormal homoclinic filaments described above. This bifurcation point turns out to be the Kovalevskaya value at which the two curves of Steklov and Bobylev meet.

5. Kovalevskaya waves

We now investigate the existence of traveling waves solutions for the Kirchhoff model. We introduce the variable $\chi = s - ct$ where c is the speed of the traveling solution. There is a remarkable scaling property (first noticed in [5]) that the Kirchhoff equations enjoy. If we consider the following change of variables:

$$\tilde{\mathbf{g}} = \mathbf{g} - c^2\mathbf{d}_3\,, \tag{5.1 a}$$

$$\tilde{\mathbf{m}} = (1-c^2)\mathbf{m} - (a+1-b)c^2k_3\mathbf{d}_3\,, \tag{5.1 b}$$

then the traveling waves solution of the Kirchhoff model are given by the solution of the following system of equations:

$$\partial_\chi \tilde{\mathbf{g}} = \mathbf{0}\,, \tag{5.2 a}$$

$$\partial_\chi \tilde{\mathbf{m}} + \mathbf{d}_3 \times \tilde{\mathbf{g}} = \mathbf{0}\,, \tag{5.2 b}$$

with the closing relation:

$$\tilde{\mathbf{m}} = (1-c^2)\,[k_1\mathbf{d}_1 + ak_2\mathbf{d}_2] + \left[b-(a+1)c^2\right] k_3\mathbf{d}_3\,. \tag{5.3}$$

Therefore, traveling waves solutions are the stationary solutions of the Kirchhoff equations with a constitutive relationship given by (5.3). The analysis is therefore straightforward and there exists localized traveling waves solutions when $\tilde{b} = 2\tilde{a}$, that is $2b = a + 2c^2(a+1)$. For these values there exists a 4 parameter family of traveling waves solutions of the form (4.6) with s replaced by χ.

6. Conclusions

The search for integrable cases of mechanical systems was once the state of the art in Mathematics. All the great mathematicians of the nineteenth century have battled wit to attach their names to them. Nowadays, the hunt for special cases of integrability seem a bit out of fashion. The interest has shifted on one side to the theory of integrable systems in pure mathematics where the notion of an explicit solution or the connection to physical models have disappeared altogether; or on the other side in applied sciences, to the theory of chaotic and dynamical systems which is, for a large part, dominated by geometric and numerical treatments. In all likelihood, a modern day Sonya Kovalevskaya would not waste effort in solving classical mechanics problems. However, we have tried to show in this paper that her contribution is still important for the understanding of mechanical problems and that the knowledge of these rare gems of ancient mathematics should not be lost. Integrable cases are lighthouses in a chaotic sea, they guide our intuition, shed lights on the turbulent flows and offer refuge for the lost minds. Like lighthouses in modern seas they seem to dwindle with time and if no effort are made to restore them future researchers might be bound to rebuild them.

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