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Painlevé analysis and normal forms theory

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Abstract

Nonlinear vector fields have two important types of singularities: the fixed points in phase space and the time singularities in the complex plane. The first singularities are locally analyzed via normal form theory, whereas the second ones are studied by the Painlevé analysis. In this paper, normal form theory is used to describe the solutions around their complex-time singularities. To do so, a transformation mapping the local series around the singularities to the local series around a fixed point of a new system is introduced. Regular normal form theory is then used in this new system. It is shown that a vector field has the Painlevé property only if the associated system is locally linearizable around its fixed points, a problem analogous to the classical problem of the center. Moreover, the connection between partial and complete integrability and the structure of local series around both types of singularities are established. A new proof of the convergence of the local Psi-series is given and an explicit method to prove the existence of finite time blow-up manifold in phase space is presented.

Keywords: Painlevé analysis; Normal forms; Singularities; Psi-series; Finite time; Blow-up; Integrability

1. Introduction

In the study of nonlinear vector fields, there are only two types of analyses that can be carried out algorithmically. The first one, originating in the classical work of Poincaré [1] and Lyapunov [2], is the traditional local analysis of a vector field around its fixed points, this is done first at the linear level by studying the linearized system and the associated (linear) eigenvalues and then at the nonlinear level by studying the local normal forms. Much can be said about the stability or lack thereof of fixed points and the resulting global dynamics. The second analysis is the analysis of scale invariant solutions which represent the asymptotic behavior of the solution around a complex-time singularity. Again, it is possible to linearize the system around such solutions and to define the associated eigenvalues and study the local solutions. Even though its origin can be traced back to Weierstrass, Hoyer and Kovalevskaya [3,4], this analysis is usually referred to as the Painlevé analysis and it has been primarily used to detect integrable systems [5,6].

In this work, we develop a normal form theory for the analysis of the solutions around their singularities. To do so, we transform the original system to a new system, the *companion system*, whose behavior around the fixed points controls the behavior of the original system around its singularities. Therefore, the local normal form analysis of

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the new system around the fixed points determines the behavior of the original system around the scale invariant solutions. Many old and new results can be readily obtained through such an analysis.

First, we show that the so-called Painlevé analysis (à la ARS [5,7]) can be reduced to an analysis of the unstable manifold of the companion system's fixed points. A vector field passes the Painlevé test if all its companion systems unstable manifolds can be analytically linearized.

Second, we show that the Painlevé property (i.e., the general solution is single-valued) actually imposes more constraints on the solution of a system than the Painlevé test actually provides (a fact already demonstrated in [8]). We show that a system has the Painlevé property only if all its companion systems can be formally linearized around their fixed points and all the Kovalevskaya exponents are integers. This requirement is analogous to the constraints imposed on a planar vector field to possess a center (formal linearizability around a fixed point with imaginary eigenvalues). We use this analogy to show that there is no finite algorithm to test for the Painlevé property (even locally).

Third, we investigate the relationship between integrability and normal forms. In order for a system to be completely integrable (i.e., the existence of $(n-1)$ first integrals for an n-dimensional vector field), both the original system and the companion systems must be formally linearizable around each of their fixed points. Moreover, all the linear eigenvalues and Kovalevskaya exponents must be rational and there must exist at least $n-1$ independent resonance relations for each set of eigenvalues.

Fourth, we show that the local series around the singularities, the so-called *Psi-series*, are always convergent. This is a direct consequence of the unstable manifold theorem applied to the companion systems fixed points.

Finally, we exploit the companion system construction to show that real series solutions around the singularities can be used to build blow-up manifolds for the original system, i.e., each point on these manifolds blow-up in a finite time.

2. Local analysis around fixed points and normal forms theory

In this section, we briefly recall the basic ingredients of local analysis of vector fields. Consider the system of differential equations

$$\dot{\mathbf{x}} \equiv \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{C}^n, \tag{1}$$

where $\mathbf{f}$ is analytic. Assume that this system has k isolated fixed points $\{\mathbf{x}^{(i)}, i=1,\dots,k\}$. For each fixed point, say $\mathbf{x}^{(i)} = \mathbf{x}_*$, introduce $\mathbf{y} = \mathbf{x} - \mathbf{x}_*$ and consider the system

$$\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}) = \mathbf{f}(\mathbf{y} + \mathbf{x}_*), \quad \mathbf{y} \in \mathbb{C}^n. \tag{2}$$

Doing so, we obtain k systems that can be analyzed locally around the origin. For each of these systems, the vector $\mathbf{g}$ can be split into two parts

$$\mathbf{g} = \mathbf{g}^{\mathrm{lin}} + \mathbf{g}^{\mathrm{nli}}, \tag{3}$$

where $\mathbf{g}^{\mathrm{lin}}$ and $\mathbf{g}^{\mathrm{nli}}$ are, respectively, the *linear* and *nonlinear* parts of the vector field. The spectrum of the Jacobian matrix $D\mathbf{g}(\mathbf{0}) = D\mathbf{g}^{\mathrm{lin}}$ defines the *linear eigenvalues* $\boldsymbol{\lambda} = \{\lambda_1, \dots, \lambda_n\}$ (where the real parts of the λ_i's are assumed to be in increasing order). The fixed point is said to be *hyperbolic* if $\Re(\lambda_i) \neq 0,\ i=1,\dots,n$.

Locally around the origin, there exists a (formal) local series solutions of the form

$$\mathbf{y}(t) = \mathbf{P}(C_1\,\mathrm{e}^{\lambda_1 t}, \dots, C_n\,\mathrm{e}^{\lambda_n t}), \tag{4}$$

where $\mathbf{P}$ is a vector of formal power series in the arguments with polynomial coefficients in t and $C_1, \dots, C_n$ are arbitrary constants. For instance, along the unstable manifold $W_u(\mathbf{0})$, we have (taking $C_i = 0$ for all i, such that $\Re(\lambda_i) \le 0$):

$$\mathbf{y}(t) = \mathbf{P}_u(C_k\, e^{\lambda_k t}, \dots, C_n\, e^{\lambda_n t}) = \sum_{\mathbf{i}, |\mathbf{i}|=1}^{\infty} \mathbf{c}_{\mathbf{i}}(t)(\mathbf{C}^u)^{\mathbf{i}}\, e^{(\boldsymbol{\lambda}^u, \mathbf{i})t}, \quad (\boldsymbol{\lambda}^u, \mathbf{i}) = \sum_{j=k}^{n} \lambda_j i_j. \tag{5}$$

This series describes, locally, the $(n - k + 1)$ parameter solutions on the local unstable manifold. The coefficients $\mathbf{c}_{\mathbf{i}}(t)$ are polynomial in t of degree less than or equal to $(\boldsymbol{\lambda}^u, \mathbf{i})$. If all $\mathbf{c}_{\mathbf{i}}$'s are constant, then the series is referred to as a *pure series* (i.e., a pure series in exponentials). If all the coefficients $\mathbf{c}_{\mathbf{i}}$ are constants for $|\mathbf{i}| \le \lambda_n$, then they are constants for all $\mathbf{i}$. The convergence of this series is guaranteed by the unstable manifold theorem [9, p. 103].

Lemma 2.1. *Let* $\mathbf{y} = \mathbf{0}$ *be a fixed point of* $\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y})$ *with linear eigenvalues* $\boldsymbol{\lambda}$. *Let* $\boldsymbol{\lambda}^u$ *be the vector of linear eigenvalues with positive real parts and* $\mathbf{y}(t; \mathbf{C}^u)$ *the series* (5). *Then, there exist* $K \in \mathbb{R}$, $t_1 \in \mathbb{R}$, $\alpha \in \mathbb{R}$, $0 < \alpha < \Re(\lambda_i^u)$, $i = k, \dots, N$ *and* δ *small enough, such that for* $|\mathbf{a}| < \delta$, $t < t_1$:

$$|\mathbf{y}(t; \mathbf{a})| < K|\mathbf{a}|\, e^{\alpha t}. \tag{6}$$

The prefactor K can be explicitly related to the norm of $\exp(A^u t)$ where A^u is the Jordan block associated with the linear eigenvalues $\boldsymbol{\lambda}^u$.

2.1. Normal forms

To discuss normal forms, it is easier to think of systems of ODEs in terms of vector fields. Let

$$\delta = \sum_{i=1}^{n} \mathbf{g}_i(\mathbf{y})\partial_{\mathbf{y}_i} \tag{7}$$

be the *vector field* associated with system (1) and let $\boldsymbol{\lambda}$ be the vector of linear eigenvalues around the origin. This vector field can be simplified by a local change of coordinates i.e, one seeks a near-identity transformation θ (in general, a formal power series), such that

$$\delta_\theta = \theta^{-1}\delta\theta \tag{8}$$

is as simple as possible (this usually means that the new vector field does not contain resonant terms, see below). The condition that θ be a near-identity transformation ensures that it is invertible and that the new vector field shares the same linear eigenvalues (the linear component of δ is invariant).

For a generic choice of $\boldsymbol{\lambda}$, the new vector field δ_θ can be chosen to be linear and the transformation θ is analytic (and unique). Three interesting things can happen when such a linearization is not possible (see [10]):

- *Resonance*. There exists at least one vector of positive integer $\mathbf{m}$ (with possibly one $m_i = -1$), such that $(\mathbf{m}, \boldsymbol{\lambda}) = 0$. Then, in general, there is no formal power series transformation θ that can linearize the vector field and the simplest form of the new vector field contains only resonant monomials, i.e., it commutes with the linear part: $[\delta_\theta, \delta_{\text{lin}}] = 0$. Moreover, if the convex hull of $\boldsymbol{\lambda}$ in the complex plane does not contain the origin, then θ is analytic and δ_θ is a polynomial vector field [11]. If the linear part is diagonal then a monomial is resonant for the jth equation if $\lambda_j = (\mathbf{m}, \boldsymbol{\lambda})$.
- *Quasi-resonance*. Let $\boldsymbol{\omega}(\mathbf{m}) = (\mathbf{m}, \boldsymbol{\lambda}) \neq 0$. If there exists an increasing sequence $\{\mathbf{m}^{(1)}, \mathbf{m}^{(2)}, \dots\}$, such that

$$\boldsymbol{\omega}(\mathbf{m}^{(i)}) \underset{i\to\infty}{\to} 0, \tag{9}$$

fast enough (this is given by a Diophantine condition on the sequence, see [12]), then either the vector field can be linearized by a divergent series, or analytically transformed to a vector field that contains all the *quasi-resonant* monomial terms (i.e., monomial of the form $\mathbf{x}^{\mathbf{m}^{(i)}}$). This situation, of great theoretical interest, can be quite complex to study (especially since multiple sequences can have such a property) and plays a central role in the stability analysis of some vector fields.

- *Nihilence*. There exists at least one first integral $I = I(\mathbf{y})$, i.e., a formal power series, such that

$$\delta I = 0. \tag{10}$$

The existence of a first integral provides additional structure to the series and among others implies the existence of resonances (see Section 7).

There is a natural relationship between the local series around the fixed point and linearizable vector fields. Let $\delta = \mathbf{f}(\mathbf{y})\partial_{\mathbf{y}}$ be the vector field associated with the system $\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y})$. This vector field can be split, in a chart independent manner, into two parts [13]

$$\delta = \delta_{\text{diag}} + \delta_{\text{nil}}, \tag{11}$$

where δ_{nil} commutes with δ_{diag}:

$$[\delta_{\text{diag}}, \delta_{\text{nil}}] = 0, \tag{12}$$

and δ_{diag} can be linearized by a formal near-identity transformation

$$\theta^{-1}\delta_{\text{diag}}\theta = \delta_{\text{lin}}, \tag{13}$$

where δ_{lin} is the linearized vector field around the origin. A vector field is *linearizable* if and only if its *nilpotent* component δ_{nil} vanishes identically.

Lemma 2.2. *The system $\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y})$ has pure local series around $\mathbf{y} = \mathbf{0}$ if and only if the vector field $\delta_{\mathbf{g}}$ is formally linearizable and the linear part δ_{lin} is semi-simple.*

Proof. First, assume that δ is formally linearizable and that $D\mathbf{g}(\mathbf{0})$ is semi-simple. Without loss of generality, we write $D\mathbf{g}(\mathbf{0}) = \text{diag}(\boldsymbol{\lambda})$. There exists a near-identity formal change of variable $\mathbf{y} = \boldsymbol{\theta}(\mathbf{z})$ that linearizes the system. In the new variable, the vector field reads: $\dot{\mathbf{z}} = \text{diag}(\boldsymbol{\lambda})\mathbf{z}$. Hence, we have $z_i = C_i\,\mathrm{e}^{\lambda_i t}$ for $i = 1, \dots, n$ and the substitution of this solution in the series $\mathbf{y} = \boldsymbol{\theta}(\mathbf{z})$ defines a pure series.

Second, assume that the system $\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y})$ has pure local series and, without loss of generality, that the Jacobian matrix $D\mathbf{g}(\mathbf{0})$ is in Jordan normal form. If the Jordan normal form is not diagonal, then the first term of the local series contains polynomials in t [14]. Indeed, the first terms of these series are the series solutions of the linearized system. We conclude that $D\mathbf{g}(\mathbf{0})$ is diagonal and δ_{lin} is semi-simple. Now, consider the pure series $\mathbf{y} = \mathbf{P}(C_1\,\mathrm{e}^{\lambda_1 t}, \dots, C_n\,\mathrm{e}^{\lambda_n t})$, where $\mathbf{P}$ is a formal power series in its arguments with constant coefficients. Since $D\mathbf{g}(\mathbf{0})$ is diagonal, $\mathbf{P}$ is of the form $\mathbf{P}(\cdot) = \mathbf{Id}(\cdot) + \mathbf{Q}(\cdot)$, where $\mathbf{Id}(\cdot)$ is the identity and $\mathbf{Q}(\cdot)$ a formal power series with no linear terms. Now, let $z_i = C_i\,\mathrm{e}^{\lambda_i t}$ for all i and consider the change of variable $\mathbf{y} = \mathbf{P}(\mathbf{z})$. This transformation is invertible since $\mathbf{P}$ is a near-identity transformation. It is also a formal change of variable that linearizes the vector field δ. □

One can also realize that if δ_{nil} is not identically zero, then the normal form in the variables $\mathbf{z}$ is not linear and at least one equation, say z_1, has at least one monomial resonant term $\dot{z}_1 = \lambda_1 z_1 + \mathbf{z}^{\mathbf{m}} + \text{h.o.t.}$, where $\mathbf{m}$ is such that $(\boldsymbol{\lambda}, \mathbf{m}) = 0$. The local series of this equation contains polynomial terms in t and so does the local series in the original variables.

Hence, the absence of resonant terms in the normal forms implies that the local solution can be expressed only in terms of exponentials (a pure solution). Conversely, the presence of resonant terms (or a non-semi-simple linear part) implies that the coefficients of the local solutions are polynomial in t. This observation is well known in physics where the presence of the so-called "secular terms" (polynomials in t) in the center manifold dynamics is known to be associated with resonances between linear eigenvalues. Note that the existence of resonance amongst the eigenvalues is not enough to create secular terms (since δ_{nil} may vanish identically), i.e., some vector fields with resonances can still be linearized.

The normal form analysis can be restricted on the unstable, stable and center manifolds. For instance, consider the unstable manifold and assume that the variables $\mathbf{y} = (\mathbf{y}^{\mathrm{s}}, \mathbf{y}^{\mathrm{c}}, \mathbf{y}^{\mathrm{u}})$ have been chosen, such that $D\mathbf{g}(0) = \mathrm{diag}(A^{\mathrm{s}}, A^{\mathrm{c}}, A^{\mathrm{u}})$, where $\mathrm{Spec}(A^{\mathrm{s,c,u}}) = \boldsymbol{\lambda}^{\mathrm{s,c,u}}$ (the vectors of eigenvalues with negative, zero and positive real parts). Then, one can find a formal transformation $\mathbf{y} = \boldsymbol{\theta}(\mathbf{z})$, such that the new vector field, in the variables $\mathbf{z}$, reads

$$\delta_{\boldsymbol{\theta}} = \mathbf{g}^{\mathrm{s}}(\mathbf{z})\partial_{\mathbf{z}^{\mathrm{s}}} + \mathbf{g}^{\mathrm{c}}(\mathbf{z})\partial_{\mathbf{z}^{\mathrm{c}}} + \mathbf{g}^{\mathrm{u}}(\mathbf{z})\partial_{\mathbf{z}^{\mathrm{u}}}. \tag{14}$$

The restriction of $\mathbf{g}$ to the unstable manifold obtained by setting $\mathbf{z}^{\mathrm{s}} = 0$ and $\mathbf{z}^{\mathrm{c}} = 0$ gives a polynomial vector field in the variables $\mathbf{z}^{\mathrm{u}}$:

$$\delta^{\mathrm{u}} = \mathbf{g}^{\mathrm{u}}(\mathbf{z}^{\mathrm{u}})\partial_{\mathbf{z}^{\mathrm{u}}}. \tag{15}$$

Moreover, the restriction of the transformation $\boldsymbol{\theta}$ to the local unstable manifold given by $\mathbf{y} = \boldsymbol{\theta}(\mathbf{z}^{\mathrm{s}} = 0, \mathbf{z}^{\mathrm{c}} = 0, \mathbf{z}^{\mathrm{u}})$ is analytic. If system (15) is linear and A^{u} semi-simple, the solution is $z_i^{\mathrm{u}} = C_i\, \mathrm{e}^{\lambda_i^{\mathrm{u}} t}$ and the analytic transformation defines again a convergent pure series.

3. Local analysis around time singularities and Painlevé analysis

Consider again an n-dimensional system of ODEs

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n, \tag{16}$$

where $\mathbf{f}$ is analytic. Assume that this system can be split into two parts

$$\mathbf{f} = \mathbf{f}^{\mathrm{up}} + \mathbf{f}^{\mathrm{down}} \tag{17}$$

in such a manner that:

1. $\mathbf{x} = \boldsymbol{\alpha}(t - t_*)^{\mathbf{p}}$ is an exact solution [1] of $\dot{\mathbf{x}} = \mathbf{f}^{\mathrm{up}}(\mathbf{x})$ with $\boldsymbol{\alpha} \in \mathbb{C}^n\, (|\boldsymbol{\alpha}| \neq 0)$ and,
2. $\mathbf{f}^{\mathrm{down}}$ is such that

$$\frac{\mathbf{f}^{\mathrm{down}}(t^{\mathbf{p}})}{t^{\mathbf{p}-\mathbf{1}}} \underset{t\to 0}{\to} 0. \tag{18}$$

Let $1/q$ be the smallest integer, such that $\mathbf{f}^{\mathrm{down}}(t^{\mathbf{p}}\mathbf{x}) = (t^{\mathbf{p}-1})\sum_{i=1}^{\infty} t^{iq}\mathbf{f}^{(i)}(\mathbf{x})$.

There is usually more than one solution $\boldsymbol{\alpha}(t - t_*)^{\mathbf{p}}$. We call a *balance* a pair $\{\boldsymbol{\alpha}, \mathbf{p}\}$ and, by extension, the corresponding decomposition (17) will be also referred to as a *balance*. Each balance corresponds to a different type

[1] $\boldsymbol{\alpha}(t - t_*)^{\mathbf{p}}$ is the vector of components $\alpha_i(t - t_*)^{p_i}$.

of behavior of the solution close to a *movable* singularity t_*. A solution in phase space can exhibit different types of behavior as it comes close enough to different fixed points. Similarly, a solution can exhibit different types of behaviors as it comes close to different movable singularities. In general, the part of the system $\mathbf{f}^{\mathrm{up}}$ that determines the balance contains the most nonlinear terms of the vector field.

We assume that there exist $k > 0$ balances $\{\{\boldsymbol{\alpha}^{(i)}, \mathbf{p}^{(i)}\}, i = 1, \ldots, k\}$ and that the vector $\mathbf{p}^{(i)}$ is rational. For each balance, say $\{\boldsymbol{\alpha}, \mathbf{p}\}$, we introduce the matrix

$$K = D\mathbf{f}^{\mathrm{up}}(\boldsymbol{\alpha}) - \mathrm{diag}(\mathbf{p}). \tag{19}$$

The eigenvalues $\{\rho_1, \ldots, \rho_n\} = \mathrm{Spec}(K)$ are the *Kovalevskaya exponents*. We complete the set of Kovalevskaya exponents by adding the value $\rho_{n+1} = q$ and define $\boldsymbol{\rho} = \{\rho_1, \ldots, \rho_n, \rho_{n+1}\}$, $\hat{\boldsymbol{\rho}} = \{\rho_1, \ldots, \rho_n\}$ (where $\rho_1 = -1$, $\rho_{n+1} = q$ and $\check{\boldsymbol{\rho}} = (\rho_2, \ldots, \rho_n)$ are ordered). The balance is said to be *hyperbolic* if $\Re(\rho_i) \neq 0,\ i = 1, \ldots, n$ and it is *principal* if $\Re(\rho_i) \geq 0\, \forall\, i > 1$. By analogy with the linear eigenvalues we define $\boldsymbol{\rho}^{\mathrm{u}}$ to be the vector of Kovalevskaya exponents with positive real parts.

3.1. Painlevé property and Painlevé tests

Locally around the singularity t_* corresponding to a balance $\{\boldsymbol{\alpha}, \mathbf{p}\}$, there exists a (formal) local series solutions of the form

$$\mathbf{x}(t) = \tau^{\mathbf{p}}\mathbf{P}(C_1\tau^{\rho_1}, \ldots, C_{n+1}\tau^{\rho_{n+1}}), \tag{20}$$

where $\tau = t - t_*$ and $\mathbf{P}$ is a vector of formal power series in the arguments with polynomial coefficients in $\log\tau$, such that $\mathbf{P}(\mathbf{0}) = \boldsymbol{\alpha}$.

A system exhibits the *Painlevé property* if its general solution is single-valued. This property is therefore a *global* property and, in general, no *local* analysis can answer the question of whether a system exhibits the Painlevé property (in the particular case of the Painlevé property, local analysis can miss essential singularities, see [15]). Nevertheless, any global property has local implications. In this case, if a system has the Painlevé property then the local solutions (even the formal ones) must be single-valued.

If, in the series (20), we consider the positive powers of τ only (i.e., taking $C_i = 0$ for all i, such that $\Re(\rho_i) \leq 0$), we have

$$\mathbf{x}(t) = \tau^{\mathbf{p}}\mathbf{P}_{\mathrm{u}}(C_k\tau^{\rho_k}, \ldots, C_{n+1}\tau^{\rho_{n+1}}) = \tau^{\mathbf{p}}\left(\boldsymbol{\alpha} + \sum_{\mathbf{i},|\mathbf{i}|=1}^{\infty} \mathbf{c}_{\mathbf{i}}(\log\tau)(\mathbf{C}^{\mathrm{u}})^{\mathbf{i}}\tau^{(\boldsymbol{\rho}^{\mathrm{u}},\mathbf{i})}\right), \quad (\boldsymbol{\rho}^{\mathrm{u}}, \mathbf{i}) = \sum_{j=k}^{n+1}\rho_j i_j. \tag{21}$$

If no logarithmic term appears (i.e., the local series is a *pure series*), then the series is a *Laurent series* when all the positive Kovalevskaya exponents are integers and a *Puiseux series* if they are rational. It is called a *Psi-series* if it is not a Puiseux series (this includes the possibility of having logarithmic terms and/or non-rational Kovalevskaya exponents). If $n - 1$ of the Kovalevskaya exponents $\hat{\boldsymbol{\rho}}$ are positive, then the series (21) is a local expansion of the general solution.

A *Painlevé test* is a procedure providing necessary conditions for the Painlevé property. The simplest such test is to check that, for each balance, the solution $\mathbf{x}(t)$, given by (21) is single-valued, i.e., a Laurent series. This *minimal Painlevé test* is the starting point of all Painlevé tests, the one used by Kovalevskaya in the rigid body motion (without actually checking the absence of logarithmic terms), the ones used by Painlevé and Gambier and the one rediscovered by Ablowitz et al. [7]. The remarkable property of this minimal test is that it can be checked in a finite number of steps (by building the series solutions up to the largest Kovalevskaya exponent).

4. Companion systems

There is an obvious parallel between the analysis of the solutions around fixed points in phase space and the analysis around the movable singularities (underlined by the analogy between the series solutions (4) and (5), and (20) and (21)). In more than one way the construction of the series (21) is similar to the construction of the solutions (5) on the unstable manifold. In order to further develop this analogy, we introduce a change of variables which transforms the local analysis around the singularities into the local analysis around the fixed point of another system. Consider a specific balance $\{\boldsymbol{\alpha}, \mathbf{p}\}$ together with the decomposition of the vector field $\mathbf{f}$:

$$\dot{\mathbf{x}} = \mathbf{f}^{\mathrm{up}}(\mathbf{x}) + \mathbf{f}^{\mathrm{down}}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{C}^n \tag{22}$$

and define the transformation $T: (\mathbf{x}, t) \to (\mathbf{X}, s)$ by

$$\mathbf{x}(t) = \tau^{\mathbf{p}} \mathbf{X}(\tau), \tag{23}$$

$$\tau = \mathrm{e}^s. \tag{24}$$

The *companion system* is then

$$X_i' = F_i(X_1, \dots, X_{n+1}), \quad i = 1, \dots, n, \qquad X_{n+1}' = qX_{n+1}, \quad \text{i.e., } X_{n+1} = \mathrm{e}^{qs}, \tag{25}$$

which we write

$$\mathbf{X}' = \mathbf{F}(X_1, \dots, X_N), \quad \mathbf{X} \in \mathbb{C}^N, \; N = n + 1. \tag{26}$$

In general, the embedding $n \to n+1$ is necessary to rewrite the new system as an autonomous vector field. However, in the particular case where the system is *weight-homogeneous* (i.e., $\mathbf{f} = \mathbf{f}^{\mathrm{up}}$), the embedding is not necessary (since $F_i (i = 1, \dots, n)$ does not depend on X_{n+1} and the last equation decouples). A companion system can be defined for each balance, therefore, there are in general k companion systems associated with a given system. Some of these systems are actually identical (for balances with equal vectors $\mathbf{p}$ but different $\boldsymbol{\alpha}$). The companion system is, for real $\mathbf{p}$, a real analytic vector field. However, in the following, we will carry out its analysis around complex fixed points (whenever $\boldsymbol{\alpha} \in \mathbb{C}^n$). The transformation to companion systems is not new, it has been used independently by different authors [16,17] in the context of the Painlevé analysis but the use of normal form analysis in this context seems new (even though similar ideas have been presented in [18,19]).

A simple Example. Consider the planar vector field

$$\dot{x}_1 = x_2, \tag{27a}$$

$$\dot{x}_2 = 4x_1^3 - 2a. \tag{27b}$$

The possible singular behaviors are: $\mathbf{p} = (-1, -2)$ with $\boldsymbol{\alpha} = (\pm\frac{1}{2}\sqrt{2}, \mp\frac{1}{2}\sqrt{2})$, i.e., $x_1 = \alpha_1\tau^{-1}, x_2 = \alpha_2\tau^{-2}$ is an exact solution of $\dot{\mathbf{x}} = \mathbf{f}^{\mathrm{up}}$ with $\mathbf{f}^{\mathrm{up}} = (x_2, 4x_1^3)^{\mathrm{T}}$ and $\mathbf{f}^{\mathrm{down}} = (0, -2a)^{\mathrm{T}}$. The first companion transformation is

$$x_1 = \tau^{-1} X_1, \tag{28a}$$

$$x_2 = \tau^{-2} X_2, \tag{28b}$$

and $\tau = t - t_* = \mathrm{e}^s$. The new companion system reads

$$X_1' = -X_2 + X_1, \tag{29a}$$

$$X_2' = 4X_1^3 + 2X_2 - 2X_3^3 a, \tag{29b}$$

$$X_3' = X_3. \tag{29c}$$

The second companion transformation $x_1 = -\frac{1}{2}\sqrt{2}\tau^{-1}X_1$, $x_2 = \frac{1}{2}\sqrt{2}\tau^{-2}X_2$ leads to the same system (since the vector $\mathbf{p}$ is the same for both balances).

5. Painlevé tests and unstable manifolds

We can now perform a local analysis of the companion system around its fixed points. By construction, there are at least two fixed points. The first one, $\mathbf{X}_0 = \mathbf{0}$, is the origin and the second one is $\mathbf{X}_* = (\boldsymbol{\alpha}, 0)$. Around the origin, the linear eigenvalues are $\mathrm{Spec}(D\mathbf{F}(\mathbf{0}) = \{-\mathbf{p}, q\}$, i.e., the eigenvalues are simply given by the exponents of the singular solution $\mathbf{x} = \boldsymbol{\alpha}\tau^{\mathbf{p}}$ together with the exponent q characterizing the non-dominant part of the vector field $\mathbf{f}^{\mathrm{down}}$.

The second fixed point is more interesting. Indeed, we find $\mathrm{Spec}(D\mathbf{F}(\mathbf{X}_*)) = \boldsymbol{\rho}$. The linear eigenvalues of the fixed point $\mathbf{X}_*$ of the companion system are the Kovalevskaya exponents of the original system around the singular solution (with the extra eigenvalue q due to the embedding). The unstable manifold of the fixed point $\mathbf{X}_*$ can be parameterized by

$$\mathbf{X}(s) = \mathbf{P}_{\mathrm{u}}(C_k\,\mathrm{e}^{\rho_k s}, \dots, C_N\,\mathrm{e}^{\rho_N s}) = \mathbf{X}_* + \sum_{\mathbf{i},|\mathbf{i}|=1}^{\infty} \mathbf{c}_{\mathbf{i}}(s)\,\mathrm{e}^{(\boldsymbol{\rho}^{\mathrm{u}},\mathbf{i})s}, \quad \Re(\rho_i) > 0, \quad i > k-1, \tag{30}$$

and $N - k + 1 = \dim(W_{\mathrm{u}}(\mathbf{X}_*))$. In terms of the original variables, we have (remember, as $\tau \to 0$, $s \to -\infty$):

$$\mathbf{x}(t) = \tau^{\mathbf{p}}\left(\boldsymbol{\alpha} + \sum_{\mathbf{i},|\mathbf{i}|=1}^{\infty} \hat{\mathbf{c}}_{\mathbf{i}}(\log\tau)\tau^{(\boldsymbol{\rho}^{\mathrm{u}},\mathbf{i})}\right), \quad \boldsymbol{\rho}^{\mathrm{u}} = (\rho_k, \dots, \rho_N), \tag{31}$$

where $\hat{c}_{\mathbf{i},j} = c_{\mathbf{i},j}$, $j = 1, \dots, n$, i.e., the local series on the unstable manifold of the companion system's fixed point is the Psi-series of the original system for the balance $\{\boldsymbol{\alpha}, \mathbf{p}\}$. This series contains as many free arbitrary constants as the number of positive Kovalevskaya exponents plus the free constant t_* corresponding to the arbitrary location of the singularity. Therefore, a solution with $n-1$ positive Kovalevskaya exponents is a local expansion of the general solution.

Every non-zero fixed point of the companion system corresponds to a possible balance $\{\boldsymbol{\beta}, \mathbf{p}\}$. Indeed, if $\bar{\mathbf{X}}$ is a fixed point, then $x_i(t) = \bar{X}_i\tau^{p_i}$ is an asymptotic solution of the original system (i.e., an exact solution of $\mathbf{f}^{\mathrm{up}}$). If $\bar{\mathbf{X}} = \mathbf{0}$, the corresponding solution is not a balance, but the origin of the original system. However, the converse is not true and if $\{\boldsymbol{\beta}, \mathbf{q}\}$ is another balance with $\mathbf{q} \neq \mathbf{p}$, then $X_* = (\boldsymbol{\beta}, 0)$ is not a fixed point of the companion system associated with the balance $\{\boldsymbol{\alpha}, \mathbf{p}\}$. Therefore, one way to analyze locally all possible balances of a given system is to find all possible decompositions of the vector field $\mathbf{f}$ and the corresponding vectors $\mathbf{p}$. To each of these decompositions corresponds a companion system. The local analysis of all non-vanishing fixed points of these companion systems provides the local analysis of all possible balances of the original system. In the following, we consider the behavior of the companion systems around a fixed point $\mathbf{X}_*$.

6. Painlevé property and the problem of the center

The Painlevé property implies that all local series are Laurent series. In particular, the minimal Painlevé test implies that the local series with ascending powers are all Laurent series. In terms of the dynamics of the companion system, we have:

Proposition 6.1. *Assume that* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ *has the Painlevé property. Then all its companion systems are such that*:

1. *The linear eigenvalues of* $\mathbf{X_0}$ *and* $\mathbf{X}_*$ *are integer*.
2. *The Jacobian matrix of the companion system at* $\mathbf{X}_*$ *is semi-simple*.
3. *The unstable manifold of* $\mathbf{X}_*$ *is analytically linearizable*.

Proof. The proof of (1) and (2) follows directly from the identification of the linear eigenvalues at $\mathbf{X_0}$ with the exponents $\mathbf{p}$ and the linear eigenvalues at $\mathbf{X}_*$ with the Kovalevskaya exponents. In order for a system to have the Painlevé property the Kovalevskaya matrix must be semi-simple with integer eigenvalues [8]. A system passes the minimal Painlevé test if all the local series solutions are Laurent series. In particular, this implies that for each balance $\{\boldsymbol{\alpha}, \mathbf{p}\}$, we have $\mathbf{p} \in \mathbb{Z}^n$, $q \in \mathbb{Z}$ and $\hat{\boldsymbol{\rho}} \in \mathbb{Z}^n$. To prove the rest of the proposition, we restrict the results of Lemma 2.2 on the unstable manifold. First, we assume that the unstable manifolds of $\mathbf{X}_*$ is linearizable and let $\mathbf{Y} = \mathbf{X} - \mathbf{X}_*$. If an unstable manifold is linearizable then it is analytically linearizable (since the eigenvalues are all in the Poincaré domain). Therefore, there exists an analytic change of variable $\mathbf{Y} = \theta^{\mathrm{u}}(\mathbf{Z}^{\mathrm{u}})$, such that the vector field for $\mathbf{Z}^{\mathrm{u}}$ is linear. If the Jacobian matrix of the companion system at $\mathbf{X}_0$ is semi-simple, the solution is $Z_i^{\mathrm{u}} = C_i\,\mathrm{e}^{\rho_i^{\mathrm{u}} s}$. The substitution of this solution in the analytic transformation $\mathbf{Y} = \theta^{\mathrm{u}}(\mathbf{Z})$ defines a (convergent) pure series, which written in terms of the original variable reads

$$\mathbf{x}(t) = \tau^{\mathbf{p}}\left(\boldsymbol{\alpha} + \sum_{\mathbf{i},|\mathbf{i}|=1}^{\infty} \mathbf{c_i}\tau^{(\boldsymbol{\rho}^{\mathrm{u}},\mathbf{i})}\right). \tag{32}$$

This series is a Laurent series if $\mathbf{p}$ and $\boldsymbol{\rho}^{\mathrm{u}}$ are integers. Hence, the linear eigenvalues of $\mathbf{X_0}$ and the positive linear eigenvalues of $\mathbf{X}_0$ are integers. □

The converse is true. If the unstable manifold of $\mathbf{X}_*$ is analytically linearizable, then the local series around the singularities with ascending powers are Laurent series. Indeed, assume that the system passes the minimal Painlevé test. Then, the local series around the singularities are all of the form (32) with integers $\mathbf{p}$ and $\boldsymbol{\rho}^{\mathrm{u}}$. This series defines the local series parameterizing the unstable manifold of the fixed point $\mathbf{X}_*$ for the companion system

$$\mathbf{X} = \mathbf{X}_* + \sum_{\mathbf{i},|\mathbf{i}|=1}^{\infty} \mathbf{c_i}\,\mathrm{e}^{(\boldsymbol{\rho}^{\mathrm{u}},\mathbf{i})s}. \tag{33}$$

By contradiction, the fact that all coefficients $\mathbf{c_i}$ are constant, implies that the unstable manifold of $\mathbf{X}_*$ is linearizable and that the Jacobian matrix of the companion system at $\mathbf{X}_0$ is semi-simple.

The existence of an analytic change of variables linearizing the dynamics of the companion system on the unstable manifold implies (i) that the Laurent solutions of the original system with ascending powers are all convergent (a fact already known from [20,21]) and (ii) that there exist locally around the singular solutions k analytic first integrals (where k is the number of positive eigenvalues). For a principal balance, this implies that locally around the singular solution $\mathbf{x} = \boldsymbol{\alpha}\tau^{\mathbf{p}}$, the system is completely analytically integrable (see next section for a converse statement).

The Painlevé property actually imposes much stronger conditions on the local structure of the companion systems' solutions.

Proposition 6.2. *If a system* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ *enjoys the Painlevé property then all its companion systems are such that*:

1. *The linear eigenvalues of* $\mathbf{X}_*$ *and* $\mathbf{X}_0$ *are integers*.
2. *The Jacobian matrix of the companion system at* $\mathbf{X}_*$ *is semi-simple*.
3. *The companion system is formally linearizable around* $\mathbf{X}_*$.

Proof. Consider the formal solutions around the singularity t_*:

$$\mathbf{x}(t) = \tau^{\mathbf{p}}\mathbf{P}(C_1\tau^{\rho_1}, \dots, C_{n+1}\tau^{\rho_{n+1}}), \tag{34}$$

where $\mathbf{P}$ is a formal power series in its arguments with coefficients polynomial in $\log(\tau)$, i.e.,

$$\mathbf{x}(t) = \tau^{\mathbf{p}}\left(\boldsymbol{\alpha} + \sum_{\mathbf{i},|\mathbf{i}|=1}^{\infty} \hat{\mathbf{c}}_{\mathbf{i}}\tau^{(\boldsymbol{\rho},\mathbf{i})}\right), \quad (\boldsymbol{\rho}, \mathbf{i}) = \sum_{j=1}^{n+1}\rho_j i_j. \tag{35}$$

As a consequence of Painlevé's $\boldsymbol{\alpha}$-method [8], a necessary condition for the Painlevé property is that the formal solution (35) is a Laurent series, i.e., $\hat{\mathbf{c}}_{\mathbf{i}}$ is constant for all $\mathbf{i}$. This also implies that both $\mathbf{p}$ and $\boldsymbol{\rho}^{\mathrm{u}}$ are integer vectors. The companion transformation maps (35) to the local series

$$\mathbf{X} = \mathbf{X}_* + \sum_{\mathbf{i},|\mathbf{i}|=1}^{\infty} \mathbf{c}_{\mathbf{i}}\, \mathrm{e}^{(\boldsymbol{\rho},\mathbf{i})s}, \quad (\boldsymbol{\rho}, \mathbf{i}) = \sum_{j=1}^{N}\rho_j i_j. \tag{36}$$

The condition for the Painlevé property is that $\mathbf{X}(s)$ is a pure series. Lemma 2.2, in turn, implies that the companion system is formally linearizable and that the Jacobian matrix of the system at $\mathbf{X}_*$ is semi-simple. □

In [8], some examples were given of systems satisfying the minimal Painlevé test but failing to have the Painlevé property (see next example). In the context of the companion systems, these examples have nonlinear normal forms. The nonlinear monomials, however, correspond to resonances between negative and positive eigenvalues or between negative resonances. The minimal Painlevé test only consider monomial corresponding to resonances between positive eigenvalues.

The last proposition implies that every companion system is formally linearizable around all its fixed points (except maybe at the origin). Moreover, since -1 is always a linear eigenvalue of $\mathbf{X}_*$, all the fixed points but the origin have resonant eigenvalues. Therefore, the problem of proving the Painlevé property is equivalent to proving that the companion systems can be formally linearized around a resonant fixed point.

In terms of normal forms, let $\Delta = \mathbf{F}(\mathbf{Y})\partial_{\mathbf{Y}}$, $\mathbf{Y} = \mathbf{X} - \mathbf{X}_*$ be the vector field of the companion system. As explained before, we split this vector field into two parts: $\Delta = \Delta_{\mathrm{diag}} + \Delta_{\mathrm{nil}}$, where $[\Delta_{\mathrm{diag}}, \Delta_{\mathrm{nil}}] = 0$ and $\theta^{-1}\Delta_{\mathrm{diag}}\theta = \Delta_{\mathrm{lin}}$. The Painlevé property implies that for each companion system

$$\Delta_{\mathrm{nil}} = 0, \quad \rho_i, p_i \in \mathbb{Z} \ \forall i. \tag{37}$$

6.1. A digression: the problem of the center

The problem of proving the Painlevé property of a given analytic vector field has a classical analogue in the theory of dynamical systems: the problem of the center. Consider a planar vector field $\delta = \mathbf{f}(\mathbf{x})\partial_{\mathbf{x}}$ field with imaginary ($\pm\mathbf{i}$) eigenvalues at $\mathbf{0}$:

$$\dot{x}_1 = x_2 + f_1(x_1, x_2), \tag{38a}$$

$$\dot{x}_2 = -x_1 + f_2(x_1, x_2). \tag{38b}$$

The problem of the center consists, for a given $\mathbf{f}$, in proving the existence of a center at the origin, i.e., the origin is a fixed point surrounded by an open sets of periodic orbits. To do so, one looks for a near-identity change of variables that removes all the nonlinear terms, i.e., δ has a center at $\mathbf{0}$ if and only if it can be formally linearized around $\mathbf{0}$:

$$\theta^{-1}\delta\theta = \delta_{\mathrm{lin}}, \tag{39}$$

which, equivalently implies $\delta_{\text{nil}} = 0$. Moreover, there exists a formal (power series) first integral (provided by θ). The main difficulty in the problem of the center resides in proving the existence of a formal linearizing power series. Indeed, there is no a priori bound on the degree of nonlinear resonant terms, i.e., if we can linearize the system up to degree N (i.e., if all monomial terms of degree less than or equal to N can be removed by polynomial changes of variables), there is no guarantee that the degree $N+1$ can also be linearized. Hence, in general, proving that a fixed point is a center is not a finite decision procedure. Conversely, proving that a fixed point is not a center is a finite decision procedure (it is enough to show that some monomials of degree N cannot be removed by formal power series change of variables) but the number of steps is not known.

This analogy shows that the Painlevé property cannot be, in general, fully tested by local analysis only. Indeed there is no general finite algorithmic decision procedure to check that $\Delta_{\text{nil}} = 0$.

6.2. An algorithm and an example

Based on the last proposition, necessary conditions for the Painlevé property can be obtained by computing the normal forms of the companion system around $\mathbf{X}_*$.

An algorithm for the Painlevé property (necessary conditions):

1. Find all possible balances $\{\boldsymbol{\alpha}, \mathbf{p}\}$. Check that $\mathbf{p} \in \mathbb{Z}^n$ for all balances.
2. For each vector $\mathbf{p}$, apply the companion transformation $\mathbf{x} \to \tau^{\mathbf{p}}\mathbf{X}$, $X_{n+1} = \tau^q$ and $\tau \to \mathrm{e}^s$ to obtain the companion system: $\mathbf{X}' = \mathbf{F}(\mathbf{X})$.
3. For each companion system check that every non-zero fixed point has integer linear eigenvalues and is formally linearizable, i.e., show that the local normal forms are linear.

The advantage of this algorithm is that it relies on the computation of normal forms, a subject of considerable studies in computer algebra. There are to date many general results and various excellent algorithms for the computation of normal forms [11,12,22–24].

An example. Consider the fourth-order equation

$$\frac{\mathrm{d}^4 x}{\mathrm{d}t^4} - x\frac{\mathrm{d}^3 x}{\mathrm{d}t^3} + 2\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = 0. \tag{40}$$

The corresponding system with $\mathbf{x} = (x, \dot{x}, \ddot{x}, \dddot{x})$ is

$$\dot{x}_1 = x_2, \tag{41a}$$

$$\dot{x}_2 = x_3, \tag{41b}$$

$$\dot{x}_3 = x_4, \tag{41c}$$

$$\dot{x}_4 = -x_1 x_4 + 2x_3 x_2. \tag{41d}$$

It is a weight-homogeneous vector field (i.e., $\mathbf{f}^{\text{up}} = \mathbf{f}$) with exponents $\mathbf{p} = (-1, -2, -3, -4)$. There is a unique balance given by $\boldsymbol{\alpha} = (-12, 12, -24, 72)$ and the Kovalevskaya exponents are $\boldsymbol{\rho} = \{-1, -2, -3, 4\}$. The minimal Painlevé test is trivially satisfied since there is a unique positive Kovalevskaya exponent. Similarly, there exists Laurent series with descending powers. However, this system does not have the Painlevé property. To see that, we apply the algorithm: first apply the companion transformation with $\mathbf{p} = (-1, -2, -3, -4)$ to obtain

$$X_1' = X_1 + X_2, \tag{42a}$$

$$X_2' = 2X_2 + X_3, \tag{42b}$$

$$X_3' = 3X_3 + X_4, \tag{42c}$$

$$X_4' = 4X_4 + X_1X_4 + 2X_3X_2. \tag{42d}$$

As expected, the linear eigenvalues around $\mathbf{X}_* = (-12, 12, -24, 72)$ are $\boldsymbol{\rho} = (-1, -2, -3, 4)$. The unstable manifold of this fixed point is analytically linearizable (since there is only one positive eigenvalue). The stable manifold is characterized by the three negative eigenvalues.

The resonance condition $\rho_i = m_1\rho_1 + m_2\rho_2 + m_3\rho_3$ leads to three possibilities: $i = 2$ and $m_2 = 1, m_1 = m_3 = 0$; $i = 3$ and $m_1 = 1, m_2 = 1, m_3 = 0$ or $m_1 = 3, m_2 = m_3 = 0$. Therefore, in order to check that the stable manifold is linearizable one has to compute the normal form to order 4. The normal form is obtained by first translating the fixed point to the origin, second diagonalizing the linear part and third applying a series of near identity transformations. To fourth order, the normal form reads

$$Y_1' = -Y_1 + 36Y_2Y_3Y_4 + \mathrm{O}(\mathbf{Y}^4), \tag{43a}$$

$$Y_2' = -2Y_2^2 - 60Y_3^2Y_4 + \mathrm{O}(\mathbf{Y}^4), \tag{43b}$$

$$Y_1' = -3Y_1 + \mathrm{O}(\mathbf{Y}^4), \tag{43c}$$

$$Y_2' = 4Y_4 + \mathrm{O}(\mathbf{Y}^4). \tag{43d}$$

The dynamics on the unstable manifold is obtained by setting $Y_1 = Y_2 = Y_3 = 0$ and is obviously linear to order 4 (hence to all order). The dynamics on the stable manifold is obtained, similarly, by setting $Y_4 = 0$ which is again linear and one concludes that the stable manifold can also be analytically linearized. In terms of the original system, it means that both local series with ascending and descending powers are Laurent series and the system passes the minimal Painlevé test. However, the system does not have the Painlevé property since the normal form is not linear. This implies that the general solution of the original system will exhibit movable logarithmic branch points.

7. First integrals

Consider $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{f}$ analytic. Assume that all nonlinear and linear eigenvalues are simple (i.e., the Jacobian matrix of the vector field around all fixed points can be diagonalized and so can the Jacobian matrix of all the companion systems around their fixed points). The function $I = I(\mathbf{x})$ is a *formal first integral* if I is a formal power series and $\delta I = 0$.

In the following, whenever we evaluate first integrals on a solutions, or assume the existence of multiple formal first integrals, we assume that there is a non-zero sector connected to the fixed points where all the formal first integrals and the formal solutions under consideration are defined. This is the case when, for instance, the first integrals are globally defined.

Theorem 7.1. *If a system has k independent formal first integrals* $I_1, \ldots, I_k$, *then*:

1. *For each fixed point* $\mathbf{x}_*$, $\exists$ *k linearly independent positive integer vectors* $\{\mathbf{m}^{(1)}, \ldots, \mathbf{m}^{(k)}\}$, *such that*

$$(\mathbf{m}^{(i)}, \boldsymbol{\lambda}) = 0, \quad i = 1, \ldots, k. \tag{44}$$

2. *For each balance* $\{\boldsymbol{\alpha}, \mathbf{p}\}$, $\exists$ *k linearly independent positive integer vectors* $\{\mathbf{m}^{(1)}, \ldots, \mathbf{m}^{(k)}\}$, *such that*

$$(\mathbf{m}^{(i)}, \check{\boldsymbol{\rho}}) = d_i \in \mathbb{Q}, \quad i = 1, \ldots, k. \tag{45}$$

Proof. (1) Since $D\mathbf{f}(\mathbf{0})$ is semi-simple, we can assume without loss of generality that the linear part of the vector field is in diagonal form. Consider $I_1, \ldots, I_k$, k independent formal first integrals and let $J_1, \ldots, J_k$ be k other formal first integrals, polynomial in $I_1, \ldots, I_k$ chosen, such that

$$J_i = \mathbf{x}^{\mathbf{m}^{(i)}} + \sum_{\mathbf{n}, |\mathbf{n}| \geq |\mathbf{m}^{(i)}|, \mathbf{n} \neq \mathbf{m}^{(i)}} a_{\mathbf{n}}^{(i)} \mathbf{x}^{\mathbf{n}}, \quad i = 1, \ldots, k, \tag{46}$$

where $\{\mathbf{m}^{(1)}, \ldots, \mathbf{m}^{(k)}\}$ are linearly independent positive integer vectors. The independence of these vectors is guaranteed by the linear independence of the first integrals' gradients. Now, let $J = \mathbf{x}^{\mathbf{m}} + \sum a_{\mathbf{n}} \mathbf{x}^{\mathbf{n}}$ be one of the first integrals and compute δJ:

$$\delta J = \mathbf{f} \cdot \partial_{\mathbf{x}} J, \tag{47a}$$

$$\delta J = (\boldsymbol{\lambda}, \mathbf{m}) \mathbf{x}^{\mathbf{m}} + \sum_{\mathbf{n}, \mathbf{n} \neq \mathbf{m}} b_{\mathbf{n}} \mathbf{x}^{\mathbf{n}}. \tag{47b}$$

Since $\delta J = 0$, we have $(\boldsymbol{\lambda}, \mathbf{m}) = 0$ and the result follows.

(2) The existence of k independent formal first integrals $I_1, \ldots, I_k$ for $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ implies the existence of k independent weight-homogeneous polynomial first integrals $J_1, \ldots, J_k$ for the weight-homogeneous system $\dot{\mathbf{x}} = \mathbf{f}^{\mathrm{up}}(\mathbf{x})$. Now, consider the companion system of $\dot{\mathbf{x}} = \mathbf{f}^{\mathrm{up}}(\mathbf{x})$ corresponding to the balance $\{\boldsymbol{\alpha}, \mathbf{p}\}$. This companion system has a fixed point located at $\mathbf{X}_* = (\boldsymbol{\alpha}, 0)$. Therefore, we apply a linear transformation $\mathbf{X} - \mathbf{X}_* = M\mathbf{Y}$, such that, in the variables $\mathbf{Y}$, the companion system is $\mathbf{Y}' = \mathbf{F}(\mathbf{Y}) = \mathrm{diag}(\boldsymbol{\rho}, 1)\mathbf{Y} + \mathbf{G}(\mathbf{Y})$, where $\mathbf{G}(\mathbf{Y})$ does not have any linear terms and $X_N = Y_N$. The fixed point $\mathbf{X} = \mathbf{X}_*$ is now $\mathbf{Y} = 0$. Let $\hat{\mathbf{Y}} = (Y_2, \ldots, Y_{N-1})$, $Z = Y_N$ and consider the first integrals $J_1, \ldots, J_k$ written in terms of the variables $\mathbf{Y}$:

$$J_i(\mathbf{Y}) = J_i(t^{\mathbf{p}}(\mathbf{X}_* + M\mathbf{Y})) = Z^{-d_i} \sum_m \left(\sum_{0 < (\mathbf{p}, \mathbf{n}) \leq d_i} a_{\mathbf{n}}^{(i)} Y_1^m \hat{\mathbf{Y}}^{\mathbf{n}} \right), \quad i = 1, \ldots, k. \tag{48}$$

Since J_i is a first integral, we have $\partial_{Y_1} J_i = 0$. Indeed, the first column vector of M is the eigenvector of eigenvalue $\rho = -1$. This eigenvector is proportional to $\mathbf{f}^{\mathrm{up}}(X_*)$. Therefore, $\partial_{Y_1} J_i$ is a sum of $\partial_{\mathbf{x}} J(\mathbf{X}_*) \mathbf{f}^{\mathrm{up}}(\mathbf{X}_*)$ and higher derivatives which all vanish identically. As before, we can choose $\hat{J}_1, \ldots, \hat{J}_k$ polynomial in $J_1, \ldots, J_k$, such that

$$\hat{J}_i = Z^{-d_i} \left(\hat{\mathbf{Y}}^{\mathbf{m}^{(i)}} + \sum_{\mathbf{n}, |\mathbf{n}| \geq |\mathbf{m}^{(i)}|} a_{\mathbf{n}}^{(i)} \hat{\mathbf{Y}}^{\mathbf{n}} \right), \quad i = 1, \ldots, k, \tag{49}$$

where $\{\mathbf{m}^{(1)}, \ldots, \mathbf{m}^{(k)}\}$ are linearly independent positive integer vectors. The condition $\delta_{\mathbf{f}^{\mathrm{up}}} J_i = 0$ then implies $\delta_{\mathbf{F}} \hat{J}_i = 0$, i.e., $(\mathbf{m}^{(i)}, \check{\boldsymbol{\rho}}) = d_i$ for $i = 1, \ldots, k$. □

As a corollary, we recover a well-known result of the theory of integrability (different versions can be found in [17,25–29]).

Corollary 7.1. *If any set of linear eigenvalues or Kovalevskaya exponents are* $\mathbb{N}$ *-independent, then the system* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ *has no formal first integral.*

The same type of result holds for algebraic first integrals. A function $I = I(\mathbf{x})$ is an *algebraic first integral* if there exists a polynomial p in $(\mathbf{x}, I)$ with coefficients in $\mathbb{C}$, such that for all solutions $\mathbf{x}(t)$ there exists a constant I, such that $p(\mathbf{x}, I) = 0$; i.e., $\delta I(\mathbf{x}) = 0$. The main difference is that the vectors $\mathbf{m}$ are now integers and the condition on $\mathbb{N}$-independence in the corollary is replaced by a $\mathbb{Z}$-independence condition. The proof is given in [30].

If the number of independent formal first integrals for an n-dimensional vector field is $k = n - 1$, the system is *completely integrable*. In this case, the local structure of the solutions is further restricted.

Theorem 7.2. *Consider a vector field* $\dot{\mathbf{x}} = \mathbf{f}(x)$. *Let* $\mathbf{x}_*$ *be a fixed point and* $\mathbf{X}_*$ *the fixed point of the companion system associated with the balance* $\{\boldsymbol{\alpha}, \mathbf{p} \in \mathbb{Q}^n\}$. *If* $\dot{\mathbf{x}} = \mathbf{f}(x)$ *is completely integrable then*:
(1) *The linear eigenvalues are rationally related* ($\exists\lambda$, *such that* $\lambda_i = q_i\lambda$ $q_i \in \mathbb{Q}$).
(2) *The Kovalevskaya exponents are rational*.
(3) $\delta = \mathbf{f}\partial_{\mathbf{x}}$ *is (formally) linearizable around* $\mathbf{x}_*$.
(4) $\Delta = \mathbf{F}\partial_{\mathbf{X}}$ *is (formally) linearizable around* $\mathbf{X}_*$.

In particular, this theorem implies that the local series around the fixed points (respectively the singularities) are pure series in exponential of t (respectively in powers of $t - t_*$), i.e., there is no polynomial in t (respectively polynomial in $\log(t - t_*)$).

Proof. (1) Since there exist $n-1$ first integrals, there exist, from Theorem 2.1, $n-1$ linearly independent integer vectors $\{\mathbf{m}^{(1)}, \dots, \mathbf{m}^{(n-1)}\}$, such that $(\mathbf{m}^{(i)}, \boldsymbol{\lambda}) = 0$. Hence, $\exists\lambda$, such that $\lambda_i = q_i\lambda$, $q_i \in \mathbb{Q}$, $i = 1, \dots, n$.

(2) In the second case, the existence of $n-1$ linearly independent integer vectors $\{\mathbf{m}^{(1)}, \dots, \mathbf{m}^{(n-1)}\}$, such that $(\mathbf{m}^{(i)}, \check{\boldsymbol{\rho}}) = d_i \in \mathbb{Q}$ implies that all the components of $\check{\boldsymbol{\rho}}$ are rational.

(3) Without loss of generality, assume that the system is written in the variables $\mathbf{x}$, such that $\mathbf{x}_* = 0$ and the linear part is diagonal. Moreover, assume it has $n-1$ first integrals, $I_1, \dots, I_{n-1}$. We can evaluate these first integrals on the general local solutions reordered in powers of t:

$$\mathbf{x}(t) = \sum_{i=1}^{\infty} \boldsymbol{\Psi}_i t^i, \tag{50}$$

where $\boldsymbol{\Psi}_i = \boldsymbol{\Psi}_i(C_1\,\mathrm{e}^{\lambda_1 t}, \dots, C_n\,\mathrm{e}^{\lambda_n t})$ is a power series in its arguments with constant coefficients. Assume, by contradiction that $\boldsymbol{\Psi}_1 \neq \mathbf{0}$. Then, to second order in t we have

$$I_i(\mathbf{x}) = I_i(\boldsymbol{\Psi}_0) + t\,\partial_{\mathbf{x}} I_i(\boldsymbol{\Psi}_0) \cdot \boldsymbol{\Psi}_1 + \mathrm{O}(t^2). \tag{51}$$

Since I_i is constant, this implies

$$I_i(\boldsymbol{\Psi}_0) = C_i, \tag{52a}$$

$$\partial_{\mathbf{x}} I_i(\boldsymbol{\Psi}_0) \cdot \boldsymbol{\Psi}_1 = 0, \quad i = 1, \dots, n-1. \tag{52b}$$

The first integrals I_i are functionally independent, therefore their gradients evaluated on $\boldsymbol{\Psi}_0$ are linearly independent. Since $\boldsymbol{\Psi}_1 \neq 0$, (52b) implies that $\boldsymbol{\Psi}_1$ is tangent to the flow, i.e., $\boldsymbol{\Psi}_1 = K(t)\mathbf{f}(\boldsymbol{\Psi}_0)$. However, since $\mathbf{x}(t)$ is a solution we have $\dot{\boldsymbol{\Psi}}_0 + \boldsymbol{\Psi}_1 = \mathbf{f}(\boldsymbol{\Psi}_0)$. Using $\boldsymbol{\Psi}_1 = K(t)\mathbf{f}(\boldsymbol{\Psi}_0)$, we have $\dot{\boldsymbol{\Psi}}_0 = (1-K)\mathbf{f}(\boldsymbol{\Psi}_0)$. This, however, is not possible since, to lowest order $\Psi_{0,i} = C_i\,\mathrm{e}^{\lambda_i t} + \cdots$, which implies $K = 0$, a contradiction. We conclude that $\boldsymbol{\Psi}_1 = 0$. The same argument applies to every order: assume that $\boldsymbol{\Psi}_i = 0$ for all $0 < i < k$, then we find $\boldsymbol{\Psi}_k = K(t)\mathbf{f}(\boldsymbol{\Psi}_0)$ and conclude that $\dot{\boldsymbol{\Psi}}_0 = (1 - kK(t))\mathbf{f}(\boldsymbol{\Psi}_0)$, which in turn implies $K(t) = 0$. Since the local solutions are pure series, we conclude from Lemma 2.2 that the fixed point is formally linearizable.

(4) Rather than evaluating the first integrals on the local solutions around the fixed point, we evaluate them on the local solutions around the singularities [29]

$$\mathbf{x}(\tau) = \sum_{i=1}^{\infty} \boldsymbol{\Psi}_i Z^i, \tag{53}$$

where $Z = \log(\tau)$. The functions $\boldsymbol{\Psi}_i$ are of the form $\boldsymbol{\Psi}_i = \boldsymbol{\alpha}\tau^{\mathbf{p}}\mathbf{P}_i(\mathbf{C}\tau^{\mathbf{p}})$, where $\mathbf{P}_i(\cdot)$ are formal power series with constant coefficients. Assume, by contradiction that $\boldsymbol{\Psi}_1 \neq \mathbf{0}$. Then, to second order in Z we have

$$I_i(\mathbf{x}) = I_i(\boldsymbol{\Psi}_0) + Z\partial_{\mathbf{x}} I_i(\boldsymbol{\Psi}_0)\cdot\boldsymbol{\Psi}_1 + \mathrm{O}(Z^2). \tag{54}$$

Since I_i is constant, this implies

$$I_i(\boldsymbol{\Psi}_0) = C_i, \tag{55a}$$

$$\partial_{\mathbf{x}} I_i(\boldsymbol{\Psi}_0)\cdot\boldsymbol{\Psi}_1 = 0, \quad i = 1,\dots,n-1. \tag{55b}$$

The same argument as used in Proof (3) then shows that $\boldsymbol{\Psi}_i = \mathbf{0}$ for all i, i.e., the local solutions are all pure series and from Lemma 2.2 the companion system is linearizable around all its fixed points. □

8. Convergence of Psi-series

Consider again, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{f}$ analytic. Assume that the balance $\{\boldsymbol{\alpha}, \mathbf{p}\}$ is hyperbolic and consider the Psi-series with ascending powers

$$\mathbf{x}(t) = \tau^{\mathbf{p}}\left(\boldsymbol{\alpha} + \sum_{|\mathbf{i}|=1}^{\infty}\hat{\mathbf{c}}_{\mathbf{i}}(\log\tau)\tau^{(\boldsymbol{\rho}^{\mathrm{u}},\mathbf{i})}\right). \tag{56}$$

We want to prove the convergence of the Psi-series for $\tau = (t - t_*)$ and the arbitrary constants $\mathbf{C}^{\mathrm{u}} = (C_k, \dots, C_n)$ small enough. In the case where the Psi-series reduce to Puiseux series (i.e., without logarithmic terms), the convergence of these series has been proven in [20,21]. In the general case, recent general results on singular analysis for PDEs by Kichenassamy and co-workers [31–33] strongly suggest that that the Psi-series are convergent in general (see also [34]). This has been successfully demonstrated on many specific examples [35–38] and, in general, for planar vector fields in [16].

To prove the convergence of the Psi-series, we consider the unstable manifold of $\mathbf{X}_*$ for the corresponding companion system

$$\mathbf{X}(s) = \mathbf{X}_* + \sum_{|\mathbf{i}|=1}^{\infty}\mathbf{c}_{\mathrm{i}}(s)\,\mathrm{e}^{(\boldsymbol{\rho}^{\mathrm{u}},\mathbf{i})s}. \tag{57}$$

The convergence of the Psi-series reduces to the convergence of the exponential series for the companion system as $s \to -\infty$. This is guaranteed by the unstable manifold theorem and we have the following theorem.

Theorem 8.1. *Let* $\{\boldsymbol{\alpha}, \mathbf{p}\}$ *be a hyperbolic balance of the system* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ *and let* $\mathbf{x}(t, \mathbf{C}^{\mathrm{u}})$ *be the local Psi-series* (56) *around the singularity* t_* *with arbitrary coefficients* $\mathbf{C}^{\mathrm{u}}$. *Then, there exist* $\epsilon \in \mathbb{R}$, $\alpha \in \mathbb{R}$, $0 < \alpha < \Re(\rho_i)$, $i = k, \dots, N$ *and* $\delta \in \mathbb{R}$, *such that for* $|\mathbf{a}| < \delta$, $0 < |t - t_*| < \epsilon$ *and* $M \in \mathbb{R}$ *sufficiently large*

$$|\mathbf{x}(t, \mathbf{a})| < M|\mathbf{a}|\tau^{\alpha}. \tag{58}$$

Proof. The proof of this theorem relies on Lemma 2.1 applied to a family of companion systems. We know that the local solution on the unstable manifold of the fixed point $\mathbf{X}_*$ is such that $|\mathbf{X}(s, \mathbf{a})| < K|\mathbf{a}|\,\mathrm{e}^{\alpha s}$ for some $s < s_1$ real. Now, in order to test the convergence of the Psi-series for $t \in \mathbb{C}$, we follow the argument in [16] and consider the family of companion systems obtained by the transformation $\tau = \mathrm{e}^{s}\,\mathrm{e}^{\mathbf{i}\theta}$, where $\theta \in [0, 2\pi)$. Taking s real and fixing θ implies that τ is the distance from the origin on a ray of angle θ with the positive real axis. For each fixed θ, we obtain a new companion system $\mathbf{F}_\theta$ for which Lemma 2.1 applies, i.e., we find for each θ a value $K_\theta \in \mathbb{R}$. Now take $M = \max\{K_\theta, \theta \in [0, 2\pi)\} \in \mathbb{R}$ and the Psi-series converges for all τ in a punctured disk of radius ϵ. □

The case of a non-hyperbolic balance (when one or more eigenvalues have zero real parts) is more complicated since it leaves the possibility of having $\log(\tau)$ terms in the first term of the expansions, clearly diverging as $\tau \to 0$. The convergence of local series associated with Kovalevskaya exponents with negative real parts cannot be established in the general case. Indeed, there is no arbitrary constant associated with the non-dominant exponent q. Hence, the local series will contain both negative powers of τ associated with negative Kovalevskaya exponents and positive powers of τ associated with q, there is therefore no ordering of the powers as $\mathbf{i}$ increases and convergence can only be guaranteed for weight-homogeneous vector fields.

9. Finite time blow-up

Consider a real analytic system of ODEs

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n. \tag{59}$$

Recall that the *general solution* of (59) is a solution that contains n arbitrary constants. The *particular solutions* are obtained form the general solution by setting some of the arbitrary constants to a given value and contain less than n arbitrary constants. They describe the evolution of restricted subsets of initial conditions. A solution depending on k arbitrary constants is denoted $\mathbf{x} = \mathbf{x}(t; c_1, \dots, c_k)$ and the general solution corresponds to $k = n$. The solution based on the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$ is $\mathbf{x} = \mathbf{x}(t; \mathbf{x}_0)$.

The system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ exhibits *finite time blow-up* if there exist $t_* \in \mathbb{R}$ and $\mathbf{x}_0 \in \mathbb{R}^n$, such that for all $M \in \mathbb{R}$, there exists an $\varepsilon > 0$ satisfying

$$|t - t_*| < \varepsilon \Rightarrow \|\mathbf{x}(t; \mathbf{x}_0)\| > M, \tag{60}$$

where $\|\cdot\|$ is any l^p norm. The blow-up is *forward* in time if $t_* > t_0$ and *backward* if $t_* < t_0$.

Equivalently, we use "$\lim_{t\to t_*} \|\mathbf{x}(t, \mathbf{x}_0)\| \to \infty$" to denote such a blow-up. There are many interesting questions related to the existence of finite time blow-up in ODEs. Among others: (1) Are there sets of initial conditions $S_0^{(k)}$ of dimension k, such that $\forall \mathbf{x}_0 \in S_0^{(k)}$, $\exists t_* \in \mathbb{R}$, such that $\|\mathbf{x}(t, \mathbf{x}_0)\| \to \infty$ as $t \to t_*$ (with $t < t_*$)? (2) Are there open sets of initial conditions with the same property? (3) What is the nature of these sets? (4) Where do blow-up occurs in phase space? Some of these questions were investigated in [39,40] in a more restrictive setting.

Here again, we use the notion of companion systems developed in the previous sections. The idea is to build a set of initial conditions on the unstable manifold of a real fixed point for the companion system and show that, under certain conditions, this set is mapped to a real set of initial conditions blowing up in the original phase space. We note that, whenever $\boldsymbol{\alpha} \in \mathbb{R}^n$, the corresponding local series solution around the singularities t_* is real (for real t_* and real arbitrary constants).

Theorem 9.1. *Consider a real analytic system* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. *Assume that this system has a balance* $\{\boldsymbol{\alpha}, \mathbf{p}\}$ *with* $(k-1)$ *positive Kovalevskaya exponents* (*excluding* $\rho_{n+1} = q$) *and let* $\boldsymbol{\beta} = (-1)^{\mathbf{p}}\boldsymbol{\alpha}$. *Then, if* $\boldsymbol{\beta} \in \mathbb{R}^n$ (*respectively* $\boldsymbol{\alpha} \in \mathbb{R}^n$) *there exists a k-dimensional manifold* S_0^k *of initial conditions leading to finite forward* (*respectively backward*) *time blow-up*.

Proof. We consider the case of backward finite time blow-up. The case of forward blow-up is obtained similarly by defining $\tau = (t_* - t)$ instead of $\tau = (t - t_*)$. Consider the companion system associated with the balance $\{\boldsymbol{\alpha}, \mathbf{p}\}$, where $\boldsymbol{\alpha} \in \mathbb{R}^n$, then $\mathbf{X}_* = (\boldsymbol{\alpha}, 0)$ is a real fixed point of a real analytic system. Let $\mathbf{X}_0 \in W_{\mathrm{u}}(\mathbf{X}_*)$ be a point on the unstable manifold of $\mathbf{X}_*$. This point is mapped by the inverse of the companion transformation to a point in the original phase space that blows up in finite time. Indeed

$$\mathbf{X}(s, \mathbf{X}_0) \underset{s\to-\infty}{\to} \mathbf{X}_* \Leftrightarrow \|\mathbf{x}(t, \mathbf{x}_0)\| \underset{t\to t_*}{\to} \infty \tag{61}$$

where $\mathbf{x}_0 = \tau_0^{\mathbf{p}} \hat{\mathbf{X}}_0 \in \mathbb{R}^n$ and $t_* = t_0 - X_{0,N} \in \mathbb{R}$, $\tau_0 = t_0 - t_*$. Now, there exists, by the unstable manifold theorem, a k-dimensional manifold of points $\mathbf{X}_0$, such that $\mathbf{X}(s, \mathbf{X}_0) \to \mathbf{X}_*$ as $s \to -\infty$. This manifold is mapped to a k-dimensional manifold of blow-up points $\mathbf{x}_0$ in the original phase space. □

In the particular case when the balance is principal and $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ is real, there exists $(n-1)$ positive Kovalevskaya exponents and there exists open sets of initial conditions leading to a finite time blow-up. When some of the Kovalevskaya exponents are zero, i.e., when the fixed point of the companion system is not hyperbolic, the situation is harder to describe. This is again due to the fact that the stability of non-hyperbolic fixed points cannot be fully determined by a linear analysis. However, there is a simple case where blow-ups occur only on some components. Indeed if, for a given balance $\{\boldsymbol{\alpha}, \mathbf{p}\}$, l components of $\boldsymbol{\alpha}$ are strictly equal to zero and $k-1$ nonlinear eigenvalues have positive real parts, then there exists a *manifold* S_0^m of dimension $m \geq k + l$ leading to finite time blow-up. Moreover, the location of the blow-up set in phase space can be obtained from the leading order behavior.

Proposition 9.1. *Consider a system* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, *let* S_0^k *be the blow-up manifold obtained in Theorem* 9.1 *and* $\mathcal{O}_{\mathrm{sign}(\boldsymbol{\beta})}$ (*respectively* $\mathcal{O}_{\mathrm{sign}(\boldsymbol{\alpha})}$) *be the orthant in phase space determined by the sign of the components of* $\boldsymbol{\beta}$ (*respectively* $\boldsymbol{\alpha}$). *Then, the forward* (*respectively backward*) *blow-up occurs in the orthant of* $\boldsymbol{\beta}$ (*respectively* $\boldsymbol{\alpha}$), *i.e.,* $\mathcal{O}_{\mathrm{sign}(\boldsymbol{\beta})} \cap S_0^k \neq \emptyset$ (*respectively* $\mathcal{O}_{\mathrm{sign}(\boldsymbol{\alpha})} \cap S_0^k \neq \emptyset$).

We now discuss the existence of finite time blow-up in the presence of first integrals. In some instances, first integrals can be used to prove directly the absence of finite time blow-up. For instance, if a two-dimensional system has a first integral $I = x_1^2 + x_2^2$, there is no possibility of finite time blow-up ($I = x_{01}^2 + x_{02}^2 = x_1^2 + x_2^2 \in \mathbb{R} \Rightarrow x_1, x_2 \in \mathbb{R}\, \forall\, t$). If, however, $I = x_1^2 - x_2^2$, then blow-up cannot be ruled out as the solutions may go to infinity in such a way that the difference of the squares remains constant. It is therefore straightforward to obtain the well-known result.

Proposition 9.2. *Let* $I = I(\mathbf{x})$ *be a first integral for the real analytic system* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. *If the level sets of I are compact, then there is no finite time blow-up* (*backward or forward*) *for open sets of initial conditions.*

How is this result related to Theorem 9.1? If $I = I(\mathbf{x}, t)$ is a first integral for the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, then there exists a first integral $I^{\mathrm{up}} = I^{\mathrm{up}}(\mathbf{x})$ for the system $\dot{\mathbf{x}} = \mathbf{f}^{\mathrm{up}}(\mathbf{x})$, where as before $\mathbf{f}^{\mathrm{up}}$ is the dominant part of the vector field with respect to the balance $\{\boldsymbol{\alpha}, \mathbf{p}\}$. Since the first integral I^{up} is constant on all solutions, it is constant on the particular solution $\mathbf{x} = \boldsymbol{\alpha}\tau^{\mathbf{p}}$, therefore $I^{\mathrm{up}}(\boldsymbol{\alpha}\tau^{\mathbf{p}}) = I^{\mathrm{up}}(\boldsymbol{\alpha})\tau^{\mathrm{d}} = 0 \Rightarrow I^{\mathrm{up}}(\boldsymbol{\alpha}) = 0$. However, if $I(\mathbf{x})$ is of definite sign, so is $I^{\mathrm{up}}(\mathbf{x})$ and the relation $I^{\mathrm{up}}(\boldsymbol{\alpha}) = 0$ cannot be satisfied if $\boldsymbol{\alpha} \in \mathbb{R}^n$. So, the fact that I^{up} is of definite sign implies that the corresponding balance $\{\boldsymbol{\alpha}, p\}$ is such that $\mathrm{Im}(\boldsymbol{\alpha}) \neq 0$. This argument provides a proof of a generalization of the previous proposition.

Proposition 9.3. *Let* $I^{\mathrm{up}} = I^{\mathrm{up}}(\mathbf{x})$ *be a first integral of a dominant part of the system* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$. *If the level sets of* I^{up} *are compact then there is no finite time blow-up for an open set of initial conditions.*

An example. We consider finite non-periodic Toda lattices with indefinite metric [41,42]. These systems are variations of the classical Toda lattice. Written in Flaschka's variables [43], the N-particle system reads

$$\dot{a}_i = s_{i+1}b_i^2 - s_{i-1}b_{i-1}^2, \quad i = 1, \dots, N, \tag{62a}$$

$$\dot{b}_i = \tfrac{1}{2}b_i(s_{i+1}a_{i+1} - s_i a_i), \quad i = 1 \dots, N-1, \tag{62b}$$

where $b_0 = b_N = a_{N+1} = 0$ and s_i is either $+1$ or -1. In the particular case where all s_i are $+1$, system (62a) and (62b) is the Toda lattice. To eliminate the explicit dependence on the signs s_i, we introduce the variables

$$x_i = s_i a_i, \quad i = 1, \dots, N, \tag{63a}$$

$$y_i = s_i s_{i+1} b_i^2, \quad i = 1, \dots, N-1. \tag{63b}$$

The new system reads:

$$\dot{x}_i = y_i - y_{i-1}, \quad i = 1, \dots, N, \tag{64a}$$

$$\dot{y}_i = y_i(x_{i+1} - x_i), \quad i = 1 \dots, N-1 \tag{64b}$$

with $y_0 = y_N = x_{N+1} = 0$. Kodama and Ye [44] used the complete integrability properties of this system to investigate the occurrence of finite time blow-up. In order to study the existence of blow-up and the dimension of the blow-up manifolds, we study all possible balances of the form

$$x_i = \alpha_i \tau^{p_i}, \quad i = 1, \dots, N, \tag{65a}$$

$$y_i = \beta_i \tau^{q_i}, \quad i = 1, \dots, N-1. \tag{65b}$$

Since the system is weight-homogeneous, we have $p_i = -1$ and $q_i = -2$ for all i. One can order the balances by the number of vanishing components of the vector $\boldsymbol{\beta}$. A combinatorial computation shows that the number of balances with k vanishing components of $\boldsymbol{\beta}$ is

$$\binom{N}{k}.$$

Hence, the total number of possible balances is equal to $2^{N-1} - 1$ (the case where all components of $\boldsymbol{\beta}$ are zero corresponds to a Taylor series and not a singular solution). The occurrence of blow-up on open sets of initial conditions can be readily computed for this problem.

Proposition 9.4. *Provided that at least one but not all* $s_i = -1$, *there exists an open set of initial conditions in* $\mathbb{R}^{2N-1}$ *leading to finite time blow-up for the system* (62a) and (62b). *In this type of blow-up only three of the* $2N-1$ *variables (two components of* **a** *and one component of* **b**) *blow-up.*

Proof. In order to have blow-up for open sets of initial conditions, one has to consider the principal balances, i.e., the balances for which all nonlinear eigenvalues have a positive or null real parts. Note that since the exponents **p**, **q** are integers, the existence of backward blow-up implies the existence of forward blow-up. Hence, we only consider backward blow-up. For system (63a) and (63b), there exists $N-1$ principal balances of the form (65a) and (65b) obtained in the following way: let j be an integer between 1 and $N-1$ and choose

$$\beta_i = -\alpha_i = \alpha_{i+1} = -1 \quad \text{if } i = j, \tag{66a}$$

$$\alpha_i = \beta_i = 0, \quad \text{otherwise}. \tag{66b}$$

All these balances are principal with Kovalevskaya exponents $\boldsymbol{\rho} = \{-1, 1(n_1\,\text{times}), 2(n_2\,\text{times}), 3(n_3\,\text{times})\}$; where $n_1 = N_1$; $n_2 = N-2, n_3 = 1$ if $j = 1$ or $j = N-1$ and $n_1 = N_1, n_2 = N-3, n_3 = 2$, otherwise. The corresponding unstable manifold of the companion system can be used to build open sets of initial conditions for the original variables. Only three of the $2N-1$ variables actually blow-up (namely, x_j, x_{j+1} and y_j, for any given $j < N$), the other variables are analytic functions of τ around the singularity. Now, in terms of the original variables $(\mathbf{a}, \mathbf{b})$, we see that, for the balances under consideration, we have $\beta_j = -1$ and $\beta_i = 0 i \neq j$. Hence, for any given vector $\mathbf{s} = (s_1, \dots, s_N)$, with at least one but not all components equal to -1, there exists an entry s_j, such that $s_j s_{j+1} = -1$. For this choice of j the corresponding balance provides a real set of initial conditions in the original phase space. □

The other balances correspond to a situation where more than three variables blow-up but on smaller dimensional manifolds. In the case where all variables blow-up at the same time, we have the following proposition.

Proposition 9.5. *The Toda system* (62a) and (62b) *exhibits blow-up in all the variables* (*backward and forward*) *if and only if N is even and the signs are alternating* (*i.e.*, $s_i s_{i+1} = -1, i = 1, \dots, N-1$). *Moreover, in the case where blow-up occurs, the blow-up manifold is of dimension N.*

Proof. Blow-up occurs in all the variables if and only if there exists a balance (65a) and (65b), such that $\alpha_i \neq 0$ for all i. The computation of such a balance shows that $\alpha_i = (N+1) - 2i$ and $\beta_i = i(i-N)$. However, for N odd, $\alpha_{(N+1)/2} = 0$, in contradiction with the assumption that $\alpha_i \neq 0$. Hence, for N odd, the variable $a_{(N+1)/2}$ does not blow-up in finite time. Since all β_i are strictly negative, we have to choose $s_i s_{i+1} = -1$ to ensure that b_i is real when y_i blows up. If N is even, the $2N-1$ Kovalevskaya exponents are $\boldsymbol{\rho} = \{-N+1, \dots, -1, 1, \dots, N\}$, i.e., there are N positive Kovalevskaya exponents and the corresponding blow-up manifold is of dimension N. □

10. Conclusions

In this paper, I have used the tools of dynamical systems to describe the local behavior of solutions around their movable complex-time singularities. The main idea is to reformulate such an analysis as a fixed point analysis of a new system. Two main tools were used in that regard, the unstable manifold theorem and normal form theory. They were used to show how the Painlevé property and the complete integrability of dynamical systems can be understood in terms of linearizability of local solutions around their fixed points. The convergence of Psi-series and the existence of finite time blow-up manifolds are also direct consequences of this construction.

It would be of great theoretical interest to bypass the construction of the companion system and to develop a normal form theory of local solutions around their singularities directly in terms of the original variables. A theory of this type has already been proposed for the perturbed Euler equations in [18].

As far as integrability theory is concerned, the nonlinear analysis of the singular solutions can be completely carried out within this framework. This provides the optimal conditions that any local analysis can achieve. However, this type of local analysis only provide necessary conditions for integrability and should be completed with a global analysis of solutions in complex-time. This analysis, à la Ziglin, is obviously considerably more difficult than the analysis presented here.

There are many intriguing possibilities regarding the existence of finite time blow-up. For instance, one could possibly have orbits blowing up both backward and forward in time. These orbits would be the equivalent of homoclinic and/or heteroclinic orbits for regular dynamical systems. As usual, the next step would be to study the possibility of splitting for these orbits. One could also use the construction given here to describe the basins of attraction of finite time singularities and eventually describe their geometry.

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