NORMAL FORMS THEORY

Many nonlinear systems can be modeled by ordinary, differential, or difference equations, and a central problem of dynamical systems theory is to obtain information on the long-time behavior of typical solutions. Because it is not possible, in general, to solve these equations explicitly, we identify particular solutions (such as equilibrium solutions or periodic solutions) and try to infer global information from the behavior of nearby solutions.

As an example, consider an equilibrium state such as the downward position of a pendulum. Looking at small perturbations of this position to identify nearby periodic orbits (corresponding to small oscillations of the pendulum) shows that the downward pendulum is indeed stable. Analysis of the inverted pendulum reveals that it is unstable, in the sense that nearby solutions do not stay close to the equilibrium position.

Mathematically, an equilibrium is a fixed point of a dynamical system, and the stability analysis is carried out by linearizing the system; that is, by replacing (close to the fixed point) the nonlinear equations by linear equations for the perturbations. The resulting linear system can be solved exactly, and the analysis of these solutions may give information about the behavior of the solutions of the nonlinear system around the fixed point. This method is extremely powerful when it works, and is the basis of much dynamical system analyses. In some cases, however, the behavior of the linear system may be entirely different than that of the nonlinear system, and no information can be obtained from the linear analysis. This implies that there is crucial information contained in the nonlinear terms.

Normal forms theory is a general method designed to extract this information. The basic idea is to compute a local change of variables to transform a nonlinear system into a simpler nonlinear system that contains only the essential nonlinear terms—those that cannot be neglected without drastically changing the nature of the system. Hopefully, the new system is either linear or can be solved explicitly.

Consider the normal form analysis of the zero fixed point of systems of two differential equations of the form

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + f_1(x_1, x_2), \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + f_2(x_1, x_2), \end{aligned} \tag{1}$$

where the dots denote time derivatives and f_1 and f_2 are nonlinear analytic functions whose Taylor expansion about the origin contains no linear term. Linear analysis of the fixed point at the origin is carried out by neglecting the linear terms and considering the linear system $\dot{x}_1 = a_{11}x_1 + a_{12}x_2,\ \dot{x}_2 = a_{21}x_1 + a_{22}x_2$. The dynamics of this system is governed by the eigenvalues λ_1, λ_2 of the matrix (a_{ij}), which is assumed to be diagonalizable. If the eigenvalues have nonvanishing real parts, then the dynamics of the original nonlinear system is qualitatively equivalent to the dynamics of the linear system and the fixed point is stable if both real parts are negative and unstable as soon as one of the real parts is negative. However, if the eigenvalues are imaginary, the behavior of the nonlinear system cannot be inferred from analysis of the linear system, and the information on the stability of the origin is contained in the nonlinear terms. For example, the origin of the system $\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 + \alpha x_1^2 x_2$ is either stable or unstable depending on the sign of α.

The first step in the normal form analysis of system (1) is to introduce a linear change of variables so it reads

$$\begin{aligned} \dot{y}_1 &= \lambda_1 y_1 + g_1(y_1, y_2), \\ \dot{y}_2 &= \lambda_2 y_2 + g_2(y_1, y_2). \end{aligned} \tag{2}$$

Second, look for a near-identity change of variables in the form of power series $y_1 = z_1 + P_1(z_1, z_2),\ y_2 = z_2 + P_2(z_1, z_2)$ that simplifies system (2). To this end, expand g_1, g_2 in power series and choose the coefficients of the series $P_1,\ P_2$ so that in the $z_1,\ z_2$ variables, the system $\dot{z}_1 = \lambda_1 z_1 + h_1(z_1, z_2),\ \dot{z}_2 = \lambda_2 z_2 + h_2(z_1, z_2)$ becomes simpler than the original system. Optimally, $h_1 = h_2 = 0$, which would provide an exact linearization of the original system. In general, however, some nonlinear terms remain after transformations.

It turns out that the ability to exactly linearize the original system is intimately connected to the eigenvalues λ_1, λ_2. If either $(n_1 - 1)\lambda_1 + n_2\lambda_2 = 0$ or $n_1\lambda_1 + (n_2 - 1)\lambda_2 = 0$ for some positive integers n_1, n_2, the eigenvalues are said to be resonant (or in resonance), and in general one of the functions (h_1 or h_2) contains resonant terms of the form $z_1^{n_1} z_2^{n_2}$. In the absence of resonance, therefore, no nonlinear terms remain and exact linearization is achieved.

If the eigenvalues are purely imaginary, z_2 is the complex conjugate of $z_1 = z$, and there are infinitely many resonance conditions. The stability of the fixed point is then decided by the first non-vanishing coefficient of h_1. As an example, consider again the system $\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 + \alpha x_1^2 x_2$. After the linear change of variable $x_1 = y_1 + y_2$, $x_2 = \mathrm{i}(y_1 - y_2)$, we have

$$\begin{aligned} \dot{y}_1 &= \mathrm{i}y_1 + (\alpha/2)\left({y_1}^2 y_2 - y_1 {y_2}^2 - {y_2}^3 + {y_1}^3\right), \\ \dot{y}_2 &= -\mathrm{i}y_2 - (\alpha/2)\left({y_1}^3 - {y_1}^2 y_2 + y_1 {y_2}^2 + {y_2}^3\right). \end{aligned} \tag{3}$$

The normal form transformation to third order reads:

$$\begin{aligned} y_1 &= z_1 + (\mathrm{i}\alpha/8)\left(2\,{z_1}^3 + 2\,z_1 {z_2}^2 - {z_2}^3\right) + \text{h.o.t.}, \\ y_2 &= z_2 + (\mathrm{i}\alpha/8)\left({z_1}^3 - 2\,{z_1}^2 z_2 - 2\,{z_2}^3\right) + \text{h.o.t.}, \end{aligned} \tag{4}$$

where h.o.t. denotes higher-order terms of degree 5 and higher. Finally, the normal form becomes

$$\begin{aligned}\dot{z}_1 &= \mathrm{i}z_1 + (\alpha/2)z_1 z_2^2 + \text{h.o.t.}, \quad z_2 = z_1^*,\\ \dot{z}_2 &= -\mathrm{i}z_2 - (\alpha/2)z_1 z_2^2 + \text{h.o.t.}.\end{aligned} \tag{5}$$

From Equations (5), $\dot{\rho} = (\alpha/2)\rho^3 + \text{h.o.t.}$ where $\rho^2 = z_1 z_2$, which implies that the origin of the initial system is stable if $\alpha \leq 0$ and unstable otherwise.

If all coefficients of the normal forms vanish identically, then the fixed point is surrounded by an open set of periodic orbit (a nonlinear center). The downward position of the frictionless pendulum is an example of a center. Note, however, that the convergence of the power series P_1, P_2 is not guaranteed, in general, and further conditions on the eigenvalues and the transformation must be satisfied.

For a general N-dimensional system of differential equations, we can compute the linear eigenvalues $\lambda_1, \ldots, \lambda_N$. If the real part of one of these eigenvalues vanishes, there is no guarantee that the dynamics of the linear system is equivalent to the dynamics of the nonlinear system close to the fixed point. Again, one can find explicit near-identity power series change of variables that simplify the original system. If $n_1\lambda_1 + n_2\lambda_2 + \cdots + n_N\lambda_N = \lambda_i$ for any i between 1 and N and for positive integers $n_1, n_2, \ldots, n_N$, then the eigenvalues are in resonance, and the ith new equation will contain some resonant nonlinear terms. This resonance relation is one of the most fundamental relations in nonlinear dynamics. It determines whether linear modes (determined by the eigenvectors of eigenvalues λ) are coupled by the nonlinear terms. It is the same resonance relation that appears in different disguises in the analysis of forced linear systems and in the resonance between frequencies in Hamiltonian and celestial mechanics. Essentially, in the absence of resonances, the system evolves following the linear modes and no interaction is possible. When resonances occur, the linear modes interact through the nonlinear terms and create complex dynamics.

Normal forms theory provide a systematic way to include the effect of the nonlinear terms. At the practical level, the method as presented tends to be rather tedious because the number of monomials that have to be taken into account grows rapidly with the dimension of the system and the degree of the normalizing transformation. There are several equivalent alternatives to the computation of normal forms, including the method of “amplitude equation” for ordinary and differential equations and the Birkhoff–Gustavson transformation for Hamiltonian systems. Nevertheless, at the conceptual level, normal forms theory is a central tool to understand the rich dynamics of nonlinear systems and this general framework can be used to study and give a rigorous foundation to many other problems beside stability.

A particularly important use of normal form theory is the theory of bifurcations. In order to identify generic bifurcations, one considers the parameters of the system $\mu_1, \ldots, \mu_M$ as additional variables satisfying trivial differential equations $\dot{\mu}_i = 0,\ i = 1, \ldots, M$ and studies the normal forms of this extended system of dimension $N + M$, revealing the nature of the bifurcation. Other applications of normal forms include the analysis of chaotic systems in systems of differential equations, the formation of patterns for partial differential equations, exponentially small effects in the splitting of separatrices, and the Painlevé theory of integrable systems.

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See also **Bifurcations; Damped driven anharmonic oscillator; Painlevé analysis**

Further Reading

Arnold, V.I. 1973. *Ordinary Differential Equations*, Cambridge, MA: MIT Press (originally published in Russian, Moscow, 1971); 3rd edition, Berlin and New York: Springer, 1992

Guckenheimer, J. & Holmes, P. 1983. *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*, New York: Springer

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