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On the definition and modeling of incremental, cumulative, and continuous growth laws in morphoelasticity

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Abstract In the theory of elastic growth, a growth process is modeled by a sequence of growth itself followed by an elastic relaxation ensuring integrity and compatibility of the body. The description of this process is local in time and only corresponds to an incremental step in the total growth process. As time evolves, these incremental growth steps are compounded and a natural question is the description of the overall cumulative growth and whether a continuous description of this process is possible. These ideas are discussed and further studied in the case of incompressible shells.

1 Introduction

We consider here the growth of an incompressible elastic body. We use the theory of finite elasticity to describe the volumetric deformation of the body under mechanical loads and stresses produced through growth. The main modeling of growth is as follows: one considers a virtual incremental geometric deformation of a body which characterizes the process of growth. This process is described locally at each point and may depend on a variety of factors such as location, time, stresses, strains, nutrient concentrations and so on. Since this deformation is purely local, it takes place irrespectively of how neighboring points grow or what the global geometry of the system is. The result being that points after deformation may overlap or create cavities. Once this virtual deformation has taken place, the integrity of the body is restored by virtue of elastic strains through an incremental elastic deformation. These two steps represent the full process of elastic growth: small growth deformation followed by a small elastic deformation. The main problem addressed in this paper is how to model the cumulative effects of such small incremental growth steps into a unique growth step followed by a single elastic deformation.

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2 Growth decomposition

The deformation of the material body is given by $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ where $\mathbf{X}$ (resp. $\mathbf{x}$) describes the material coordinates of a point in the reference (resp. current) configuration of a body $\mathcal{B}_0(\mathbf{X}, t)$ (resp. $\mathcal{B}$). The main postulate in morphoelasticity first described in Rodriguez et al. (1994) is that the deformation gradient $\mathbf{F}(\mathbf{X}, t)$ can be decomposed (see Fig. 1) into a product of a growth tensor $\mathbf{G}(\mathbf{X}, t)$ with an elastic tensor $\mathbf{A}(\mathbf{X}, \mathbf{t})$ so that

$$\mathbf{F}(\mathbf{X}, t) = \mathbf{A}(\mathbf{X}, t) \cdot \mathbf{G}(\mathbf{X}, t). \quad (1)$$

This decomposition is similar to the one used in the theory of finite elasticity when other anelastic effects, such as plasticity, are included (Lee 1969; Maugin 2003); other proposals to model growth can be found in Norris (1998) and Gleason and Humphrey (2004). While this decomposition is conceptually clear and appealing, its time-dependence and actual application needs to be further explained. The decomposition is instantaneous, that is, it applies continuously with respect to time and as $\mathbf{G}$ evolves in time, $\mathbf{A}$ follows. However, in typical biological growth or physical swelling, there are actually four important time scales, the elastic time scale τ_e of elastic wave propagation, the viscoelastic time of relaxation τ_v, the time scale associated with external loading τ_l, and the growth time scale τ_g. Implicit in the decomposition is the statement that the growth time scale can be separated from the elastic time scale. Moreover, while this is not implied and there may be regimes of interest when growth time scales become comparable with other time scales, it is reasonable to assume that growth time scales are larger than other relevant time scales, so that

$$\tau_e \ll \tau_v \ll \tau_l \ll \tau_g. \qquad (2)$$

The main idea underlying this scale ordering is that as growth takes place on a slow time scale, the elastic and viscoelastic responses of the material take place on much shorter time scales (associated with the speed of sound and the viscous times, respectively) and for time smaller than τ_g the material is in elastic static equilibrium. Therefore, the only time considered in the process is the growth time t associated with the evolution law for the growth tensor:

$$\dot{\mathbf{G}} \equiv \frac{\mathrm{d}\mathbf{G}}{\mathrm{d}t} = \mathcal{H}(\mathbf{G}, \mathbf{A}, \mathbf{T}, \ldots; \mathbf{X}, t). \qquad (3)$$

A priori, the growth tensor can be a function of the stress tensor $\mathbf{T}$, the deformation tensor $\mathbf{A}$, the material position, the time, external loads, or other fields such as nutrient concentration or temperature (see below for further discussion on the constitutive laws for $\mathcal{H}$).

3 Incremental growth

Now, consider an incremental time step Δt such that $\tau_v \ll \Delta t \ll \tau_g$. The one-step Euler method for (3) yields

$$\mathbf{G}(t + \Delta t) = \mathbf{G}(t) + \Delta t \mathcal{H}(t). \qquad (4)$$

This relation defines an incremental growth step $\mathbf{G}(t + \Delta t) - \mathbf{G}(t) = \mathbf{G}_{\text{inc}}$ for which the decomposition (1) holds, that is if we denote the incremental elastic response by $\mathbf{A}_{\text{inc}}$, the deformation gradient after one incremental step Δt is given by

$$\mathbf{F}_{\text{inc}} = \mathbf{A}_{\text{inc}} \cdot \mathbf{G}_{\text{inc}}. \qquad (5)$$

At each step, for a given $\mathbf{G}_{\text{inc}}$, one computes the elastic strains and stresses necessary for mechanical equilibrium and the new growth increment is computed from

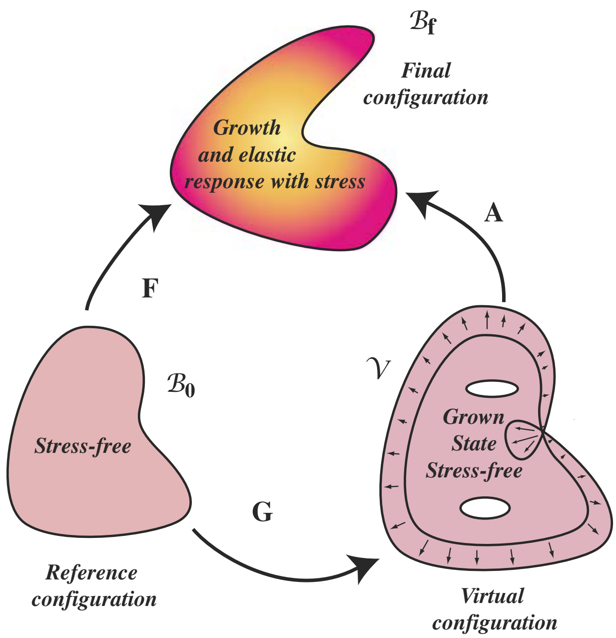

Fig. 1 The decomposition of finite morphoelasticity. Starting from a reference configuration $\mathcal{B}_0(\mathbf{X}, t)$, the deformation gradient $\mathbf{F}(\mathbf{X}, t)$ is the product of a growth tensor $\mathbf{G}$ with an elastic deformation tensor $\mathbf{A}(\mathbf{X}, t)$. The intermediate configuration is referred to as a virtual configuration since compatibility and integrity cannot be ensured

these strains. To see how the evolution is achieved, consider the first growth step. Before growth, the material is in a *natural configuration* that is, it is stress-free and we denote by $\mathbf{G}_1$ the first incremental step computed from (4) with $t = 0$ and $\mathbf{G}(0) = \mathbf{1}$ (see Fig. 2). It is possibly a function of the position $\mathbf{X}$ but not of the deformation gradient or the stresses since elastic responses

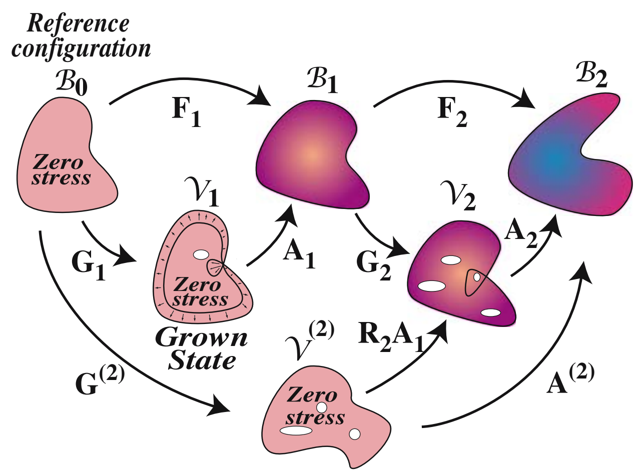

Fig. 2 Cumulative growth with two steps, the superscript (k) denotes the total cumulative deformation after k incremental steps, whereas a subscript i denotes the incremental step taking place after the $(i-1)$th step

have not yet taken place. To compute the elastic strain tensor $\mathbf{A}_1$, we assume that the response of the material (in a natural configuration) is given by a response function $\mathcal{T}$ such that the Cauchy stress tensor is for a general deformation gradient $\mathbf{A}$ given by

$$\mathbf{T} = \mathcal{T}(\mathbf{A}). \tag{6}$$

We will further assume that the material is hyperelastic and there exists a strain energy function $W = W(\mathbf{A})$ from which the stress–strain relation follows, that is, in the case of our deformation gradient $\mathbf{A}_1$:

$$\mathbf{T}_1 = J_1^{-1}\mathbf{A}_1 \cdot W_{\mathbf{A}_1} - p_1\mathbf{1}. \tag{7}$$

In this relation, $\mathbf{T}_1$ is the Cauchy stress in $\mathcal{B}_1$, $W_{\mathbf{A}_1}$ the derivative of $W(\mathbf{A_1})$ w.r.t. $\mathbf{A}_1$, $J_1 = \det(\mathbf{A}_1)$ represents the change of volume due to the elastic deformation. If the material is incompressible then $J_1 = 1$ and p_1 is the hydrostatic pressure. Otherwise, the material is compressible and $p_1 = 0$. For this discussion, we will only consider the incompressible case and set $J_1 = 1$; the compressible case, while more computationally intensive, does not alter our argument on growth laws. Since we have viscous relaxation taking place on time scales faster than growth times, there is no time dependence for the elastic part of the process and the strains and stresses are determined from the equation for mechanical equilibrium

$$\nabla_{\mathbf{x_1}} \cdot (\mathbf{T}_1) = 0, \tag{8}$$

where the divergence is taken with respect to $\mathbf{x}_1$ in the current configuration $\mathcal{B}_1$. The equation is supplemented with boundary conditions. For instance, if the body is loaded by fluid pressure P, the boundary condition is given by the Cauchy stress in the normal direction $\mathbf{n}$ of the boundary

$$\mathbf{T}_1 \cdot \mathbf{n} = -P\mathbf{n} \text{ on } \partial\mathcal{B}. \tag{9}$$

We can now perform the second growth step and compute $\mathbf{G}_2$ based on the strains and stresses ($\mathbf{A}_1$ and $\mathbf{T}_1$) found in the configuration $\mathcal{B}_1$ (see Fig. 2). However, this configuration is not stress-free and we cannot directly compute the stresses in $\mathcal{B}_2$ from a relation such as (7). Since the material is pre-stressed, the constitutive relation now depends on $\mathbf{T}_1$. The general problem of computing the stresses under loading of a pre-stressed body has been thoroughly analyzed by Hoger and collaborators in a series of papers (Hoger 1986, 1993; Johnson and Hoger 1993, 1995) and is discussed briefly in Appendix A. For our particular problem, the residual stress in configuration $\mathcal{B}_1$ is associated with a deformation gradient $\mathbf{A}_1$ from the natural configuration $\mathcal{V}_1$ and therefore, there are no issue associated with the definition of a natural configuration and the total deformation gradient is simply given by

$$\mathbf{F}^{(2)} = \mathbf{F}_2 \cdot \mathbf{F}_1 = \mathbf{A}_2 \cdot \mathbf{G}_2 \cdot \mathbf{A}_1 \cdot \mathbf{G}_1. \tag{10}$$

The problem is now to write $\mathbf{F}^{(2)}$ as a product of a growth tensor $\mathbf{G}^{(2)}$ from a reference configuration with an elastic tensor $\mathbf{A}^{(2)}$ from the grown virtual configuration $\mathcal{V}^{(2)}$ to the final configuration $\mathcal{B}_2$. That is, we need to identify correctly, the elastic deformation from the growth deformation. The problem is that the growth term can introduce a rotation of the principal axis and we must therefore remove this rotation. To do so, we must unload the configuration $\mathcal{V}_2$ to remove stress and obtain a stress-free, possibly incompatible, configuration $\mathcal{V}^{(2)}$. Following (Hoger et al. 2004), this is performed by requiring that the unloading is performed along the principal directions of the Cauchy stress tensor $\mathbf{T}_1$. The stress in $\mathcal{V}_\in$ is obtained from the deformation $\mathbf{A}_1$ followed by a possible rotation of the principal axis due to $\mathbf{G}_2$. To isolate the rotation in $\mathbf{G}_2$, we use the polar decomposition theorem to write it as a product of a symmetric tensor and an orthogonal tensor

$$\mathbf{G}_2 = \mathbf{V}_2 \cdot \mathbf{R}_2. \tag{11}$$

Therefore, the mapping from $\mathcal{V}^{(2)}$ to $\mathcal{V}_2$ is simply given by $\mathbf{R}_2 \cdot \mathbf{A}_1$ (see Fig. 2). Therefore, we identify the growth tensor

$$\mathbf{G}^{(2)} = \mathbf{A}_1^{-1} \cdot \mathbf{R}_2^{\mathrm{T}} \cdot \mathbf{G}_2 \cdot \mathbf{A}_1 \cdot \mathbf{G}_1, \tag{12}$$

where we have used the identity $(\mathbf{R}_2 \cdot \mathbf{A}_1)^{-1} = \mathbf{A}_1^{-1} \cdot \mathbf{R}_2^{-1} = \mathbf{A}_1^{-1} \cdot \mathbf{R}_2^{\mathrm{T}}$. Similarly, the elastic tensor is

$$\mathbf{A}^{(2)} = \mathbf{A}_2 \cdot \mathbf{R}_2 \cdot \mathbf{A}_1. \tag{13}$$

The invertibility of $\mathbf{A}_i$ is ensured by the fact that we consider small growth increments, which implies that the elastic relaxation $\mathbf{A}_i$ is a near-identity tensor whose deviation from identity is controlled by the step size. The stress in the configuration $\mathcal{B}_2$ is then given by

$$\nabla_{\mathbf{x_2}} \cdot (\mathbf{T}_2) = 0, \tag{14}$$

where

$$\mathbf{T}_2 = \mathbf{A}^{(2)} \cdot W_{\mathbf{A}^{(2)}} - p_2\mathbf{1}, \tag{15}$$

$W_{\mathbf{A}^{(2)}}$ being the derivative of $W(\mathbf{A}^{(2)})$ with respect to $\mathbf{A}^{(2)}$. The equation for the stress is solved with given boundary conditions which may include loadings.

4 Cumulative growth

It is now straightforward to generalize the previous results to k successive incremental deformations of the form $\mathbf{A}_i \cdot \mathbf{G}_i$. We use a superscript index (k) to denote a state after k incremental deformations. For instance, the total deformation is given by the product (see Fig. 3)

$$\mathbf{F}^{(k)} = \mathbf{A}_k \cdot \mathbf{G}_k \cdot \mathbf{A}_{k-1} \cdot \mathbf{G}_{k-1} \cdots \mathbf{A}_2 \cdot \mathbf{G}_2 \cdot \mathbf{A}_1 \cdot \mathbf{G}_1, \quad (16)$$

which we rewrite as a product $\mathbf{B}^{(k)} = \mathbf{A}^{(k)} \cdot \mathbf{G}^{(k)}$ where

$$\mathbf{G}^{(k)} = \left(\mathbf{A}^{(k-1)}\right)^{-1} \cdot \mathbf{R}_k^T \cdot \mathbf{G}_k \cdot \mathbf{A}^{(k-1)} \cdot \mathbf{G}^{(k-1)}, \quad (17)$$

$$\mathbf{A}^{(k)} = \mathbf{A}_k \cdot \mathbf{R}_k \cdot \mathbf{A}^{(k-1)}. \quad (18)$$

The stress in the final configuration is computed as before. From a computation standpoint, the strain tensors obtained at step $k+1$ are obtained from the previous ones by evaluating $\mathbf{G}_{k+1}$ based on the tensors in the configurations $\mathcal{B}_k$, introducing an unknown tensor $\mathbf{A}_{k+1}$ and defining iteratively the tensors $\mathbf{G}^{(k+1)}$ and $\mathbf{A}^{(k+1)}$.

Further progress can be made if growth and elastic tensors commute. This happens for instance when there is a basis in which the tensors are diagonal. This is the case, for instance when radial deformations are considered in a spherical geometry, or in a cylindrical geometry. It encompasses most of the cases considered in the literature and is the starting point of a stability analysis. Therefore, we consider the case when

$$\mathbf{A}_i \cdot \mathbf{G}_j = \mathbf{G}_j \cdot \mathbf{A}_i, \quad \text{for all } i, j, \quad (19)$$

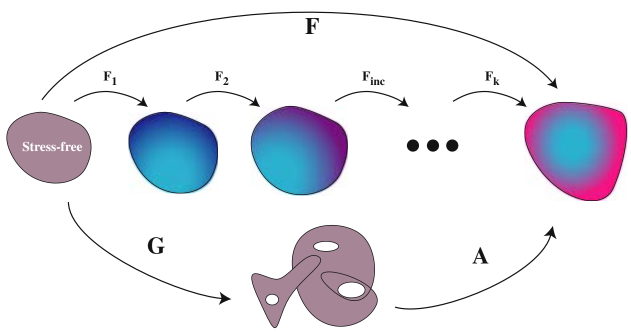

Fig. 3 Computation of deformation for a pre-stressed body. First a stress-free configuration must be obtained from which the total loading is obtained as a composition of the two deformation gradients

and the elastic tensors now takes the simpler form

$$\mathbf{A}^{(k)} = \mathbf{A}_k \cdot \mathbf{A}_{k-1} \cdots \mathbf{A}_2 \cdot \mathbf{A}_1, \quad (20)$$

$$\mathbf{G}^{(k)} = \mathbf{G}_k \cdot \mathbf{G}_{k-1} \cdots \mathbf{G}_2 \cdot \mathbf{G}_1. \quad (21)$$

5 Constitutive laws

Before we study some simple growth laws, it is of interest to review what is known and what has been proposed for the functional form of the growth rate.

There have been early on attempts to use the morphoelasticity formalism to model simple situations and understand the effect of growth and the feedback due to stress. These include the following cases:

Constant growth The simplest choice for $\mathbf{G}$ is to consider a constant tensor. A constant diagonal tensor $\mathbf{G}$ has been used in spherical geometry by Hoger and coworkers (Chen and Hoger 2000; Klisch et al. 2001). This case is interesting since analytical results can be obtained corresponding to small increments and explicit values of residual stress computed for growth without loading. In Ben Amar and Goriely (2005), the stability of such growing shell is considered.

Position dependent growth Many growth processes depend on the location in the material. This effect is sometimes referred to as *differential growth* to indicate that some parts of a tissue grows faster than others. In morphoelasticity, it implies that $\mathbf{G}$ is a function of either $\mathbf{X}$ or $\mathbf{x}$. Both situations are of interest. In the first case, growth is a function of material points $\mathbf{X}$ in the reference configuration and this dependence assumes that the material is made out of points that grow at different rates and keep growing differentially as time goes by. In the second case, the ability of a tissue to grow depends on its location at any given time. This is the case, for instance, when cell reproduction depends on the availability of some nutrients that diffuse through the boundary. At any given time, the amount of nutrient may be described by the distance to the boundary as in the case in the growth of spheroids in tumor experiments. Both cases will be considered in the spherical geometry in the following section. The stability analysis of differentially growing shells was considered in Goriely and Ben Amar (2005).

Stress-dependence It has been recognized experimentally and theoretically in many systems (such as aorta, muscles and bones) that one of the main biomechanical regulators of growth is stress (Hsu 1968; Rodriguez et al. 1994; Taber 1995, 1998; Fung 1995; Taber and Eggers 1996; Rachev 1997). It has even been suggested that stresses on cell walls play the role of a pacemaker for the collective regulation of tissue growth (Shraiman 2005).

Accordingly, the growth rate tensor should be a function of the Cauchy stress tensor which could also vary according to the position of tissue elements in the reference configuration. The study of the coupling between stress and growth remains largely unexplored and no general relationship for such coupling has been proposed (see, however, the discussion in Fung 1995).

6 Modeling incremental growth

Growth is an incremental process and its cumulative effect can be computed by following the procedures outlined in the previous sections, but an actual computation of cumulative growth is in most cases impossible. From a theoretical perspective, this situation is not satisfactory since it does not provide a tractable way to discuss parameter changes or the effects of coupling with other fields. Moreover, for the computation of stability properties, a finite deformation solution is required. Therefore, it is advantageous to model the cumulative growth deformation by a tensor $\mathbf{G}$ that captures the relevant incremental growth process over long periods of time (or, equivalently, over large changes of volume).

To gain some insight in issues involved in computing and modeling cumulative growth, we consider radially symmetric deformations of a growing shell under pressure. Furthermore, we restrict our attention to two cases of incremental growth where the incremental growth tensor is a function of the position:

Case I The growth tensor $\mathbf{G}_{\text{inc}}$ is isotropic but a function of the radial position R in the *reference* configuration, that is $\mathbf{G}_{\text{inc}} = g_{\text{inc}}(R)\mathbf{1}$. For simplicity, we further assume that there is no growth at the inner boundary $\mathbf{G}_{\text{inc}}(A) = 1$. In this case, further analytic progress can be made since the growth function g_{inc} at the kth step is a function of the original reference configuration and we have

$$g(R)_{\text{cum}} = g_{\text{inc}}^k. \tag{22}$$

This case is trivial in the sense that once incremental growth is given, so is the cumulative growth. A simpler form can be obtained by noting that since $g(A) = 1$, we can write $g_{\text{inc}}(R) = 1+\epsilon f(R-A)$, where $\epsilon = 1/k$ defines the size of each incremental growth step and $f(0) = 0$. Taking $k \to \infty$, we have an exact continuous version of the cumulative growth simply given by

$$g(R) = \exp(f(R-A)). \tag{23}$$

Case II We now consider the case where the growth tensor $\mathbf{G}_{\text{inc}}$ is isotropic but a function of the radial position r in the *current* configuration, that is $\mathbf{G}_{\text{inc}} = g_{\text{inc}}(r)\mathbf{1}$. Again, we assume that there is no growth at the inner boundary $\mathbf{G}_{\text{inc}}(a) = 1$. In many ways this simple problem already contains the complexity inherent in any growth laws since the position $r = r(R)$ of a material point depends on the boundary stresses. Therefore, by assuming that incremental growth is a function of the current configuration, we have an implicit dependence on the stress tensor which needs to be computed at each iteration. Since no closed form is available, even for simple choice of $\mathbf{G}_{\text{inc}}$, we have to find a suitable fit to model the effect of cumulative growth and suitable for stability analysis, and write a *continuous* description of growth in time (or equivalently as a function of a suitable bifurcation parameter such as volume, or boundary position).

Explicitly, the growth function g_{inc} at the kth step is now $g(R) = \prod_{i=1}^{k} g_{\text{inc}}(r_i)$ where r_i is the current configuration after the i-th deformation. However, the i-th deformation r_i, depends on the elastic strains which has to be solved at each step through the boundary conditions on the Cauchy stress. To do so, let $r_i = r_i(R)$ be the radial position of the material point of initial radial coordinate R after i incremental steps and we denote $r_i' = \frac{\mathrm{d}r_i}{\mathrm{d}R}$ so that the cumulative deformation gradient in the usual spherical coordinates after i steps is

$$\mathbf{F}_i = \operatorname{diag}\left(r_i', \frac{r_i}{R}, \frac{r_i}{R}\right). \tag{24}$$

Similarly, let

$$\mathbf{A}_i = \operatorname{diag}(\alpha_i^{-2}, \alpha_i, \alpha_i), \tag{25}$$

where we have used the incompressibility condition that imposes $\det(\mathbf{A}_i) = 1$ to express the three diagonal elements $\lambda_{i1}, \lambda_{i2}, \lambda_{i3}$ in terms of a single variable $\alpha_i = \lambda_{i2} = \lambda_{i3}$. For simplicity, we assume that the elastic shell is made out of a neo-Hookean material , that is $W = \mu(\lambda_{i3}^2 + \lambda_{i3}^2 + \lambda_{i3}^2 - 3)$. Let $a_i = r_i(A)$ and $b_i = r_i(B)$ be the radii in the current configuration. The incompressibility condition is $\det(\mathbf{A}_i) = \det(\mathbf{F}_i\mathbf{G}_i^{-1}) = 1$ implies $\det(\mathbf{F}_i) = \det(\mathbf{G}_i)$, that is

$$\frac{r_i' r_i^2}{R^2} = g_{\text{inc}}^3. \tag{26}$$

This last equation can be integrated explicitly to determines both the deformation

$$r_i^3 = a_i^3 + 3\int_{a_{i-1}}^{r_{i-1}} g_{\text{inc}}^3(\rho)\rho^2 \mathrm{d}\rho, \tag{27}$$

and the strain $\alpha_i = r_i/(g(R)R)$, up to the value of a_i that is obtained from the boundary conditions on the Cauchy stress. The radial component of the Cauchy stress t_1 is a solution of (8), whose only non-vanishing component reads

$$\frac{\partial t_1}{\partial r} + \frac{2}{r}(t_1 - t_2) = 0, \tag{28}$$

where $t_2 = T_{22} = T_{33}$ is the hoop stress. To obtain a closed equation for the radial stress t_1, one first solve the constitutive relation ship for t_2 as a function of t_1

$$t_1 = \alpha_i^{-2}\frac{\partial W}{\partial \lambda_{1i}} - p, \tag{29}$$

$$t_2 = \alpha_i \frac{\partial W}{\partial \lambda_{2i}} - p. \tag{30}$$

and substitute the result in (28) to obtain

$$\frac{\partial t_1}{\partial r_i} = \frac{\alpha_i}{r_i}\partial_{\alpha_i}\widehat{W}, \tag{31}$$

where $\widehat{W} = W(\alpha^{-2}, \alpha, \alpha)$. Furthermore, one can express t_1 as a function of the material variable R and obtain the differential equation

$$\frac{\partial t_1}{\partial R} = \frac{\partial_{\alpha_i}\widehat{W}}{R\alpha_i^2}, \tag{32}$$

where (27) is used to express $\alpha_i = \alpha_i(R)$. This last equation can be readily integrated to obtain

$$t_1(R) = \int_A^R \frac{\partial_{\alpha_i}\widehat{W}}{R\alpha_i^2}\mathrm{d}R. \tag{33}$$

The boundary conditions are $t_1(A) = t_1(B) = 0$. Once the radial stress is known, the deformation is completely determined and the hoop stress is given by $t_2 = t_1 + \frac{\alpha_i}{2}\partial_{\alpha_i}\widehat{W}$. It is important to realize that the computation of the Cauchy stress must be performed from the unloaded reference configuration since there is no simple law for the increase in stress of a preloaded configuration. That is, to compute the stress from a preloaded configuration, a completely unloaded configuration must be found and the total strain between the unloaded configuration and the new loaded configuration must be computed.

We can now iterate numerically the process and recompute at each stage the growth tensor needed for the next iterate

$$g_{\mathrm{cum}}(R) = \prod_{i=1}^{k} g_{\mathrm{inc}}(r_i). \tag{34}$$

To analyze the effect of cumulative growth, we use a simple, linear incremental growth law that is, $g_{\mathrm{inc}}(r_i) = 1+\mu(r_i - a_i)$, ($a_i = r_i(A)$ is the radial position of the inner boundary after $i-1$ steps). As an example, we choose $A = 1$, $B = 2$ and μ such that the volume increases at each step by 1% up to a total volume increase of 275%. The graph of $g_{\mathrm{cum}}(R)$ is shown on Fig. 4 and the general trend observed is one of a stretched exponential. However, it is notoriously difficult (and somewhat arbitrary) to fit a curve with a stretched exponential and important features may be lost in the process. The main problem is that during growth points closer to the outer boundary grow faster than the ones close to the inner boundary and accordingly the variable $r(R)$ is stretched out. Therefore a natural choice to express cumulative growth is in the current configuration.

As can be seen in Fig. 5, the growth function does not have exponential behavior and can be easily fitted by a function of the form $g(r) = 1 + h(r - a)$ where h is analytic and $h(0) = 0$. Both linear and quadratic fits are shown to capture the effect of growth and these functions can be used for the continuous description of the growth process with a reduced number of parameters. This is the modeling of cumulative growth that has been used for stability analysis of a differentially growing or shrinking shell (Goriely and Ben Amar 2005).

7 Conclusions

In this paper we have addressed a simple question usually overlooked in the discussion of elastic growth,

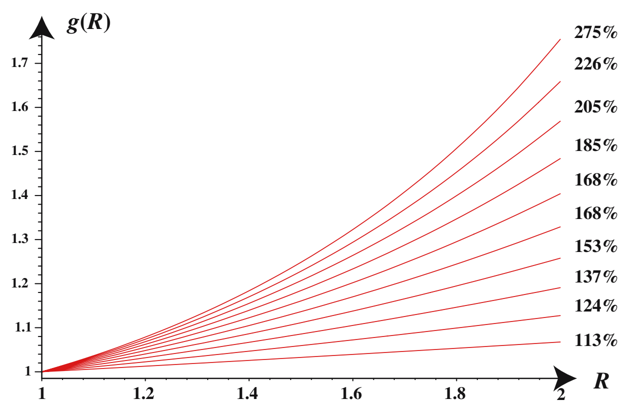

Fig. 4 The cumulative growth function in the reference frame $g = g(R)$. We consider a growing elastic shell of initial radii $R = 1$ and $R = 2$. Growth is linear in the current radius and chosen such that the inner shell experiences no growth $g(R = 1) = 1$ and the total increment in volume is 1% at each step. The cumulative growth curves are shown for various values of volume increases. Note the marked super-exponential behaviour of $g(R)$

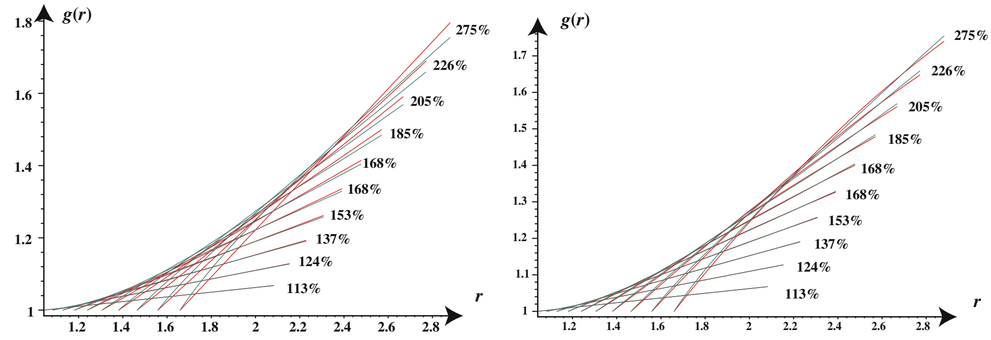

Fig. 5 The cumulative growth as computed on Fig. 4 but now viewed in the current frame, that is, $g = g(R(r))$. Since the current configurations expands with growth the cumulative function $g(r)$ can be fitted either with a linear (*left*) or quadratic fit (*right*)

namely, how to best analyze and represent the cumulative effect of incremental growth steps. This question has certainly been answered implicitly in various studies but to the best of our knowledge no discussion appear in published scientific papers. There are two main issues with the problem of cumulative growth. First, growth and elastic relaxation have to be described in terms of an unstressed configuration, otherwise stresses cannot be computed. While the theory is transparent in the case of one incremental step, it is not quite as clear for repeated steps where a virtual unstressed configuration needs to be defined. In the case where growth and elastic relaxation commute, the situation is much simpler and further analytical progress can be made. The second issue is a computational one. The cumulative effect of growth and relaxation requires the computation of the total growth and deformation from the unstressed states and depend at each step on the boundary conditions. Clearly, there is no hope to obtain a closed-form solution for this problem (except in simple cases as shown here) and modeling cumulative growth in the current configuration may be a way to circumvent this problem. Hopefully more powerful tools such as the renormalization group could be applied to the problem to obtain long-time estimate and closed-form approximations and henceforth offer a more satisfactory solution.

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Appendix: Residual stress

The main idea to obtain the stresses from a residually stressed material under loading is to find a natural configuration and compute the total strains from the natural configuration to the final configuration with loads by composition of deformation gradient (see Fig. 6). Let $\mathcal{B}_r$, a body with residual stress. That is, it supports a stress field $\mathbf{T}_r$ with zero traction at the boundary:

$$\nabla \cdot \mathbf{T}_r = 0, \quad \mathbf{T}_r.\mathbf{n} = 0 \quad \text{on } \partial\mathcal{B}_r, \tag{35}$$

where the divergence is taken in $\mathcal{B}_r$. Let $\mathbf{T} = \mathcal{T}(\mathbf{A})$ be the constitutive relation in a natural configuration. Then, assuming the deformation gradient $\mathbf{A}_r$ from the residually stressed body to a natural configuration is well defined (and this is a delicate point), the stress $\mathbf{T}_f$ in the final configuration is given by

$$\mathbf{T}_f = \mathcal{T}(\mathbf{A}_f \cdot \mathbf{A}_r), \tag{36}$$

which, expressed in terms of the deformation from $\mathcal{B}_r$ to $\mathcal{B}_f$ and the residual stress $\mathbf{T}_r$, is

$$\mathbf{T}_f = \mathcal{T}(\mathbf{A}_f \cdot \mathcal{T}^{-1}(\mathbf{T}_r)), \tag{37}$$

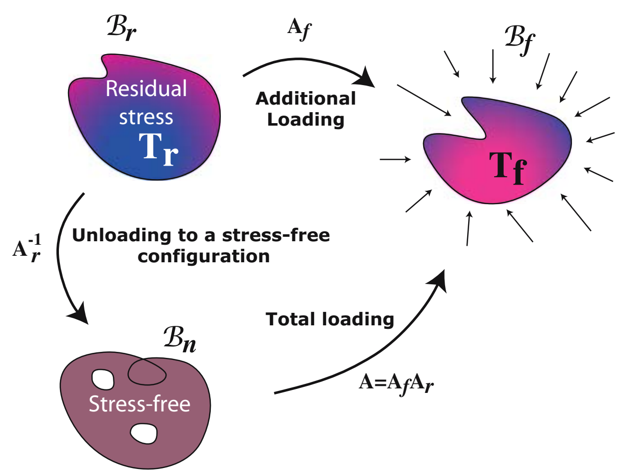

Fig. 6 Computation of deformation for a pre-stressed body. First a stress-free configuration must be obtained from which the total loading is obtained as a composition of the two deformation gradients

where we have assumed that $\mathcal{T}$ is invertible. In the case considered in this paper, invertibility is guaranteed by the fact that only incremental (near-identity) deformations are considered. Further progress can be achieved for particular material symmetry group and explicit forms for constitutive relations can be given.

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