6

Morphoelasticity - A theory of elastic growth

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6.1   Introduction

Growth is the process by which a body increases in size through the addition of mass. In biological systems, growth can serve a number of different purposes, and occurs in many different forms. Growth may be restricted to particular locations on the body. In particular, tip growth is often found in microscopic filamentary systems (Gooday and Trinci, 1980; Howard and Valent, 1996; Goriely and Tabor, 2003). Surface growth and accretion refers to the deposition of material on the surface of a body - this type of growth is found in the formation of horns, teeth, and seashells (Skalak and Hoger, 1997; Thompson, 1992). In volumetric growth, on the other hand, growth occurs throughout the bulk of the body. This is common in the growth of hearts, tumors, and arteries (Taber and Humphrey, 2001; Cowin, 2004).

Continuum mechanics and nonlinear elasticity provide a natural framework to study growth. Of foremost importance is capturing the correct relationship between growth and elasticity. The basic idea is that in a growth process, the deformation of a body can be due to both a change of mass, and an elastic response. This concept, first put on mathematical terms by Rodriguez et al. (Rodriguez, Hoger and McCulloch, 1994), states that the deformation tensor can be decomposed into a growth tensor instructing different parts of the body how to add mass locally and an elastic tensor which captures the elastic reorganization necessary to ensure integrity of the body.

A key benefit to studying growth through continuum mechanics is that it provides direct access to stresses. The relationship between stress and growth is a key concept in the theory of elastic growth. Stress may induce growth, for instance a body may grow until a homeostatic "target" stress is reached. On the other hand, growth can induce residual stress, in particular when the body is inhomogeneous or growing differentially. Residual stresses are stresses that persist in the body even after all body forces have been removed. They are known to play an important role in the functioning of a body. For instance, residual stress is key in the regulation of the transmural stress in arteries (Humphrey, 2003), and is also known to occur in tree and plant tissues (Vandiver and Goriely, 2008), leading to tension or compression woods. A particularly spectacular example of these internal stresses is found in the "Wapas" trees in French
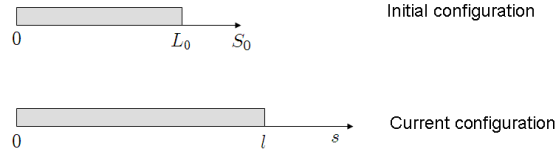
Fig. 6.1: One-dimensional deformation of a rod.

Guiana. Due to residual stress, these trees explode when cut, giving them the nickname "killing trees" (Détienne and Thiel, 1988). More generally, the generation of residual stress is connected to the question of how growth can alter the properties of a material (Goriely and Ben Amar, 2005; Ben Amar and Goriely, 2005).

This chapter is structured as follows. We begin with a 1D theory of growth, providing several examples to illustrate the basic concepts and some of the questions that arise in morphoelasticity. Then, we develop the full 3D theory from first principles. The fundamental concept, the decomposition of the deformation tensor, is derived in a manner different from any appearing in the literature. We conclude with several applications and future challenges.

6.2   1D growth

In this section we develop the theory of elastic growth in one dimension. Consider a rod of length L_0 in its initial reference configuration which undergoes a deformation so that it is of length l in the current configuration. Let S_0 describe position along the rod in the initial configuration and s position in the current configuration - see Figure 6.1. We define the geometric stretch

$$\lambda = \frac{\partial s}{\partial S_0}.$$ (6.1)

In general $\lambda = \lambda(S_0)$, and λ is constant for a uniform stretch. Also, $\lambda > 1$ corresponds to elongation and $\lambda < 1$ to a reduction. We next consider several classes of deformation.

6.2.1   Pure elastic deformation

In a pure elastic deformation, $\lambda = \alpha$, where α is the *elastic stretch*. Letting σ denote the stress, the characteristics of the deformation are determined by a constitutive relationship between σ and α. For example, "Hooke's Law" for a linear material gives $\sigma = E(\alpha - 1)$, where E is the Young's modulus. Solving for α and inserting in (6.1), the deformation is described by

$$s = \left(\frac{\sigma}{E} + 1\right) S_0,$$ (6.2)

from which we can write $l = \left(\frac{\sigma}{E} + 1\right) L_0$. In this case, the length increases linearly with the stress. A more accurate description is given by the non-linear neo-Hookean relationship $\sigma = \mu(\alpha^2 - \alpha^{-1})$, where $\mu > 0$ is the elastic modulus.

6.2.2   Pure growth

A pure growth is characterized as having no elastic response. Here, $\lambda = \gamma$, where γ describes the growth. Whereas in the previous section α describes a pure elastic *stretch*, γ describes the *addition* or *removal* of material. If $\gamma > 1$ there is growth while $\gamma < 1$ indicates shrinking or resorption of material. In general, the material may undergo growth or resorption at different times and at different rates, and a constitutive relation is needed. For pure growth, the constitutive law is of the form

$$\frac{\partial \gamma}{\partial t} = G(\gamma, s, S_0). \tag{6.3}$$

In the simple case of uniform growth, $G = 1$, which gives $\gamma = t$ and $s = S_0 t$. Exponential growth is captured by $G = k\gamma$. Then $s = S_0 e^{kt}$ and the length of the rod grows as $l(t) = L_0 e^{kt}$. Alternatively, suppose that growth only occurs towards the tip. For an exponentially growing tip, we take

$$\frac{\partial \gamma}{\partial t} = G(\gamma, s) = \begin{cases} k\gamma & 0 < s < a \\ 0 & a < s < l \end{cases} \tag{6.4}$$

Equation (6.4) is coupled with $\frac{\partial s}{\partial S_0} = \gamma$ and initial conditions $s(0) = S_0$ and $\gamma(0) = 1$. The difficulty computationally is that $s = s(S_0, t)$ is a moving interface. To understand the behavior of the growing tip, consider a discrete numeric process. At time t_i, the rod will grow according to $\gamma_i = e^{kt_i}$. How much does the rod grow in one time step, from t_i to t_{i+1}? Letting s_n be the current configuration at the nth step, we have

$$e^{kt_i} = \gamma_i = \frac{\partial s_{i+1}}{\partial S_0} = \frac{\partial s_{i+1}}{\partial s_i} \frac{\partial s_i}{\partial S_0} = \frac{\partial s_{i+1}}{\partial s_i} e^{kt_{i-1}}. \tag{6.5}$$

This implies that $\frac{\partial s_{i+1}}{\partial s_i} = e^{k(t_i - t_{i-1})} = e^{k\Delta t}$. Thus the region $0 < s_i < a$ grows by a factor $e^{k\Delta t}$, meaning that the rod extends by an amount $\Delta l = a(e^{k\Delta t} - 1)$. The important thing to note is that this is independent of the actual time, i.e. the rod extends by the same amount at every time step. Even though the tip is growing exponentially, the length of the rod is only increasing *linearly*. As another example, we present a toy model for tumor growth (see more models of this type in the contribution *Mechanical Aspects of Tumour Growth* by L. Preziosi in this volume). The setup is depicted in Figure 6.2. Let $r(R_0, t)$ give the radial position for a solid sphere in the current configuration, where R_0 is the radial position in the initial configuration, and let $b(t)$ be the radius in the current configuration. The tumor is surrounded by a bath of nutrient, which is transmitted to the tumor through diffusion with uptake. We suppose that growth is exponential but proportional to the concentration of the nutrient. Hence, growth is modelled by

$$\begin{aligned} \frac{\partial \gamma}{\partial t} &= k\gamma u(r) \\ \frac{\partial r}{\partial R_0} &= \gamma. \end{aligned} \tag{6.6}$$
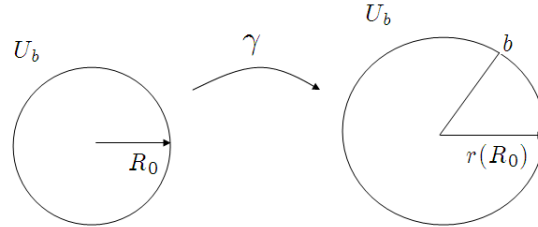
Fig. 6.2: Setup for the tumor growth toy model.

Here $u(r)$ is the concentration of the nutrient and satisfies

$$\frac{\partial u}{\partial t} = \frac{D}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) - Q, \tag{6.7}$$

where D is the diffusion coefficient and Q is the nutrient uptake (assumed to be constant). The outer boundary condition for $u(r,t)$ is $u(b,t) = U_b$, that is the bath provides a constant nutrient supply to the outer surface. The other boundary condition, and the growth behavior, depends on the size of the tumor. If b is small enough so that $u(r)$ can diffuse all the way to the center of the sphere, the boundary condition is $0 \le u(0) < \infty$. For some b_{cr}, the diffusion will satisfy $u(0) = 0$. For $b > b_{cr}$, the boundary condition is $u(a) = 0$, where $a > 0$ is the smallest radius to which the nutrient can diffuse. This is shown schematically in Figure 6.3.

For $b < b_c$, i.e. when the tumor is "small", the entire sphere grows and so growth is exponential. After this initial phase, growth is restricted to a spherical shell and so is equivalent to the tip growth problem discussed above. Thus, as a function of time, the size of the tumor $b(t)$ increases exponentially at first and then transitions to a linear rate.

We have only provided a qualitative description of the behavior of the growing tumor. As an exercise left for the reader, it is informative to derive a differential equation for the evolution of $b(t)$ and determine *as $t \to \infty$, the size of the region $b-a$?* This corresponds to the so-called "penetration length" of the nutrient.

6.2.3   Growth with elastic response

In an elastic body, growth is subject to an elastic response. The fundamental assumption of morphoelasticity, to be justified later, is that the geometric stretch $\lambda = \frac{\partial s}{\partial S_0}$ is the product of an elastic term α and a growth term γ. That is,

$$\lambda = \alpha\gamma, \tag{6.8}$$

where α satisfies some constitutive relationship with the stress, $H(\alpha, \sigma) = 0$, and γ satisfies a growth equation $\partial_t\gamma = G(\alpha, \gamma, \sigma)$.
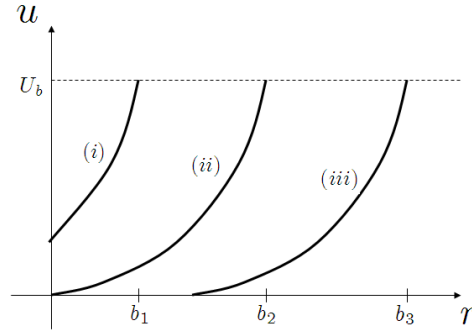
Fig. 6.3: The concentration of the nutrient $u(r)$ as a function of r for different sized tumors. In (i), the tumor radius b_1 is small enough for the nutrient to diffuse all the way to the center, and growth is exponential. In (ii), the radius is a critical value b_{cr} such that $u(0) = 0$. In (iii), $b > b_{cr}$ – here growth only occurs in the region where $u(r) > 0$ and so the growth is linear.

To give an example, consider a rod growing between two walls. We take the growth to be linear and uniform, so that $\gamma = 1 + t$, and use the linear Hookean model for the elastic relationship: $\sigma = E(\alpha - 1)$. Due to the wall constraint, we must have $\lambda = 1$. This implies $\alpha = 1/\gamma$, from which we obtain

$$\sigma = -E\left(\frac{t}{1+t}\right). \tag{6.9}$$

Essentially, as the rod grows, a compressive stress builds up to contain the rod in the fixed space. Observe from (6.9) that as $t \to \infty$, $\sigma \to -E$. This is problematic in that it states that only a finite stress is developed in compressing an infinite rod. The problem is due to the linearity assumption. If instead the non-linear neo-Hookean relationship $\sigma = \mu\left(\alpha^2 + \alpha^{-1}\right)$ is used, we obtain

$$\sigma = \mu\left(\frac{1}{1+t^2} - t + 1\right). \tag{6.10}$$

For the non-linear model, there is nothing that slows down the growth and $\sigma \to -\infty$ as $t \to \infty$.

To better understand the relationship between growth and elasticity, it is useful to consider a two-step process, pictured in Figure 6.4. Starting from the initial, stress-free configuration, in Step 1 the rod *grows* to a configuration which is stress free but not compatible because it does not fit within the walls. Hence this configuration is called a virtual configuration. Step 2 is the elastic response, which may be thought of as the problem of fitting the rod back in between the walls. This step introduces stress, and maps the rod from the virtual configuration to the current configuration.

Initial configuration

$\sigma = 0$

Current configuration.

$\sigma \neq 0$

λ

γ

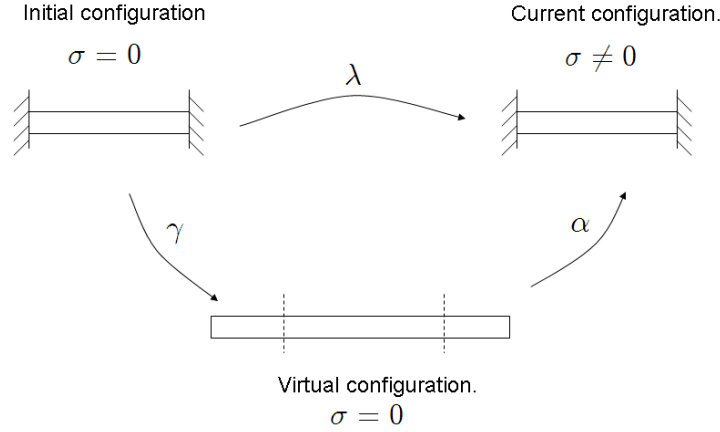α

Virtual configuration.

$\sigma = 0$

Fig. 6.4: Decomposition of the deformation into a growth step which leads to an incompatible, virtual configuration, and an elastic response step, which fits the rod back into a physically compatible state and introduces stress.

In many situations, the growth of a body depends on the stress level within the body. For instance, the body may function best at a particular stress, and grow until this target stress is reached. This situation can be modelled by

$$\frac{\partial \gamma}{\partial t} = k\gamma(\sigma - \sigma^*) \tag{6.11}$$

where σ^* is the target stress or *homeostatic stress*. Consider the rod between two walls geometry and a neo-Hookean material with $\sigma = \mu\left(\alpha^2 - \alpha^{-1}\right)$. As before, $\gamma = 1/\alpha$, and (6.11) may be recast in terms of α as

$$\frac{\partial \alpha}{\partial t} = \alpha\sigma^* - \mu(\alpha^3 - 1). \tag{6.12}$$

Setting the right hand side to zero, define α^* as the solution to $\alpha\sigma^* = \mu(\alpha^3 - 1)$, the equilibrium elastic stretch.

Assuming that the initial configuration is stress free, $\alpha(0) = 1$. If $\sigma^* > 0$, $\alpha^* > 1$ and so α will increase, limiting to α^* as $t \to \infty$. This corresponds to γ decreasing from 1 to $\gamma^* = 1/\alpha^*$. In this case, the target stress is tensile, and the rod contracts until the target stress is reached. On the other hand, if $\sigma^* < 0$, the target stress is compressive. Here, $\alpha^* < 1$, and so α decreases to α^*; that is, the rod grows until the proper compression is reached.

6.3   3D growth

In this section we develop the full 3D theory of morphoelasticity. Let $\mathbf{X}$ be the position vector for a body $\mathcal{B}_0$ before deformation, and $\mathbf{x}$ the position vector for the body in
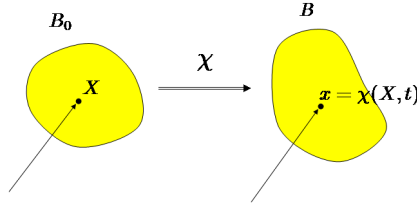
Fig. 6.5: Deformation χ from reference configuration $\mathcal{B}_0$ to current configuration $\mathcal{B}$.

the current configuration at time t, denoted $\mathcal{B}$. This is pictured in Figure 6.5. The deformation of the body is described by the function χ, so that $\mathbf{x}(t) = \chi(\mathbf{X}, t)$. Then the deformation gradient tensor is $\mathbf{F} = \mathrm{Grad}\chi$, the gradient taken with respect to the reference variables.

Without growth the motion of the body is governed by the following equations:

Conservation of mass

$$\dot{\rho} + \rho \mathrm{div}\mathbf{v} = 0 \tag{6.13}$$

Balance of linear momentum

$$\rho \dot{\mathbf{v}} = \mathrm{div}\mathbf{T} + \rho \mathbf{b} \tag{6.14}$$

Balance of angular momentum

$$\mathbf{T}^T = \mathbf{T} \tag{6.15}$$

Here ρ is the density, $\mathbf{v} \equiv \dot{\mathbf{x}} = \frac{\partial \mathbf{x}}{\partial t}$ is the velocity of a material point, $\mathbf{T}$ is the Cauchy stress, and $\mathbf{b}$ represents any body forces. Note that div is the divergence operator in the current configuration.

We now derive the corresponding equations with added growth. To do so, we will need to make use of the following transport formulas: (derived in, e.g., (Ogden, 2003))

$$\frac{d}{dt} \int_\Omega \phi \, dv = \int_\Omega (\dot{\phi} + (\mathrm{div}\mathbf{v})\phi) \, dv \tag{6.16}$$

$$\frac{d}{dt} \int_\Omega \mathbf{u} \, dv = \int_\Omega (\dot{\mathbf{u}} + (\mathrm{div}\mathbf{v})\mathbf{u}) \, dv \tag{6.17}$$

where ϕ and $\mathbf{u}$ are scalar and vector fields associated with the moving body. Assume that growth can occur through volumetric growth, captured by the growth rate function $\gamma(\mathbf{x})$, or by the flux of material through the boundary, described by the vector $\mathbf{\Pi}(\mathbf{x})$. Balance of mass, applied to an arbitrary volume element Ω yields

$$\frac{d}{dt} \int_\Omega \rho \, dv = \int_\Omega \gamma \, dv + \int_{\partial\Omega} \mathbf{\Pi} \cdot d\mathbf{s}. \tag{6.18}$$

Application of the divergence theorem and the transport formula (6.16), and using the fact that Ω was arbitrary, we localize the mass equation

$$\dot{\rho} + \rho \mathrm{div}\mathbf{v} = \gamma + \mathrm{div}\mathbf{\Pi}. \tag{6.19}$$

Balance of linear momentum is expressed as

$$\frac{d}{dt}\int_\Omega \rho\mathbf{v}dv = \int_{\partial\Omega}\mathbf{T}\cdot d\mathbf{a} + \int_\Omega \rho\mathbf{b}\,dv + \int_\Omega \gamma\mathbf{v}\,dv + \int_{\partial\Omega}\mathbf{\Pi}\otimes\mathbf{v}\,d\mathbf{s}. \tag{6.20}$$

Use of the divergence theorem on the terms integrated over $\partial\Omega$ as well as the transport formula (6.17) leads to

$$\int_\Omega \left(\rho\dot{\mathbf{v}} + \dot\rho\mathbf{v} + (\text{div}\mathbf{v})\mathbf{v}\right)\,dv = \int_\Omega \left(\text{div}\mathbf{T} + \rho\mathbf{b} + \gamma\mathbf{v} + \text{div}(\mathbf{\Pi}\otimes\mathbf{v})\right)\,dv. \tag{6.21}$$

Inserting the expression for $\dot\rho$ from (6.19) and using the arbitrary nature of Ω, (6.21) simplifies to

$$\rho\dot{\mathbf{v}} = \text{div}\mathbf{T} + \rho\mathbf{b} + (\mathbf{\Pi}\cdot\nabla)\mathbf{v}. \tag{6.22}$$

Lastly, defining $\tilde{\mathbf{T}} := \mathbf{T} + \mathbf{\Pi}\otimes\mathbf{v}$, angular momentum is balanced if $\tilde{\mathbf{T}}$ is symmetric.

To simplify the situation, it is assumed that growth processes take place on slow time scales so that we can neglect fluxes through the boundary and their momenta. This is the slow growth assumption of morphoelasticity. It enables us to set $\mathbf{\Pi} = \mathbf{0}$. Dropping all terms with $\mathbf{\Pi}$ and comparing to equations (6.13)-(6.15), the only apparent difference in the governing equations with added growth is the addition of the function γ in (6.19), that is the mass equation becomes

$$\dot\rho + \rho\text{div}\mathbf{v} = \gamma. \tag{6.23}$$

Note also that $\tilde{\mathbf{T}} = \mathbf{T}$, so that the Cauchy stress is symmetric. To close the system, a constitutive law is needed to relate the stress to the deformation, i.e. a relation of the form $\mathbf{T} = H(\mathbf{F})$. We will assume that the material is hyperelastic. In the absence of growth, the relation is

$$\mathbf{T} = J^{-1}\mathbf{F}\frac{\partial W}{\partial\mathbf{F}} - p\mathbf{1} \tag{6.24}$$

where W is the strain energy function and $J = \det\mathbf{F}$. If the material is incompressible, $J = 1$ and p is the hydrostatic pressure; for a compressible material $p = 0$. However, this relation requires the initial body $\mathcal{B}_0$ to be *unstressed*. This raises the question: what if we don't know the unstressed initial configuration? Suppose we only know the body at time t in configuration $\mathcal{B}$, a stressed state. How do we develop a response function theory for a stressed material? In answering this question we will develop the theory of morphoelasticity.

As a first step, we remove any external or body forces from $\mathcal{B}$, that is *unload the body*. Let $\mathcal{B}_r$ be this unloaded configuration, and denote χ_l as the map from $\mathcal{B}_r$ to $\mathcal{B}$. It is important to note that $\mathcal{B}_r$ is not stress free since residual stresses may still exist within the body. There are several possible approaches at this point. One is to develop a constitutive relation that includes the residual stress field. That is, if $\mathbf{T}^r$ is the stress in $\mathcal{B}_r$, one could try to find a relation $\mathbf{T} = H(\mathbf{F}, \mathbf{T}^r)$. Another possibility, outlined in (Ogden, 2003), is to add the residual stress field, i.e. define the stress to be the sum of the residual stress and the stress due to loading: $\mathbf{T} = \mathbf{T}^r + \mathbf{T}^l$.

A third option, and the one we will expand upon here, is to find a description of the residually stressed material in terms of volume element growth. The approach is
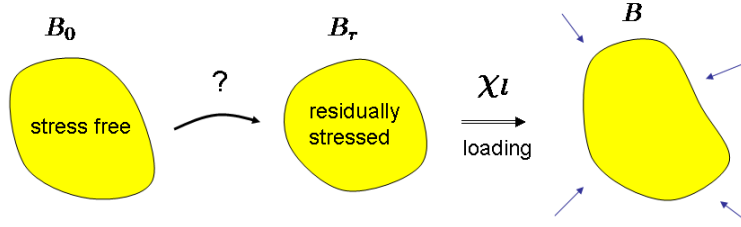
Fig. 6.6: Conceptual schematic for the theory developed here. A stress free reference configuration $\mathcal{B}_0$ is sought, such that in the deformation from $\mathcal{B}_0$ to $\mathcal{B}_r$ residual stress develops due to growth of volume elements.

to define a stress free reference configuration $\mathcal{B}_0$ and build a map from $\mathcal{B}_0$ to $\mathcal{B}_r$ where residual stress is induced in the body due to growth. This is depicted in Figure 6.6.

Conceptually, the idea is to repeatedly *cut* the body $\mathcal{B}_r$, thereby relieving the residual stress, until the body consists of a (possibly infinite) collection of stress free "components". These components together comprise a stress free *virtual configuration*. By keeping track of the deformation due to the "cuts" and reversing the directionality, a deformation tensor can be defined which takes the unstressed virtual configuration to the fully stressed current configuration. The final step then is to introduce a tensor which takes the connected and stress free state $\mathcal{B}_0$ to the virtual configuration.

Mathematically, the idea relies upon Signorini's Theorem, which states that the *mean stress* of a simply connected body which is in mechanical equilibrium and without external loads, vanishes identically, i.e.

$$\bar{\mathbf{T}} := \frac{1}{v} \int_\Omega \mathbf{T} \, dv = \mathbf{0} \tag{6.25}$$

for any volume element Ω with volume v. Signorini's Theorem along with the continuity of the stress $\mathbf{T}$ implies that there exists a point $\mathbf{x}^* \in \Omega$ such that $\mathbf{T}(\mathbf{x}^*) = \mathbf{0}$. Now, consider an arbitrary point $\mathbf{x} \in \mathcal{B}_r$. We start to cut the body and follow the portion of the body which contains $\mathbf{x}$, denoted $\mathcal{B}_r^n$ (see Figure 6.7). Define

$$d_n := |\mathbf{x} - \mathbf{x}_n^*| \tag{6.26}$$

where $\mathbf{T}(\mathbf{x}_n^*) = \mathbf{0}$. Thus, d_n is the distance between $\mathbf{x}$ and the (not necessarily unique) stress free point $\mathbf{x}_n^* \in \mathcal{B}_r^n$ after the nth cut. As $n \to \infty$, $d_n \to 0$, and thus

$$\lim_{n \to \infty} \mathbf{T}(\mathbf{x}) = \mathbf{0}.$$

The body is now completely unstressed at $\mathbf{x}$. Repeat this process for all points in $\mathcal{B}_r$ and we are left with an *unstressed virtual configuration*, which we call $\mathcal{B}_v$. To make sense of $\mathcal{B}_v$, consider that for each point $\mathbf{x}$ in $\mathcal{B}_r$ there is a map which takes the tangent space at that point to the tangent space in $\mathcal{B}_v$; in this sense $\mathcal{B}_v$ is not really
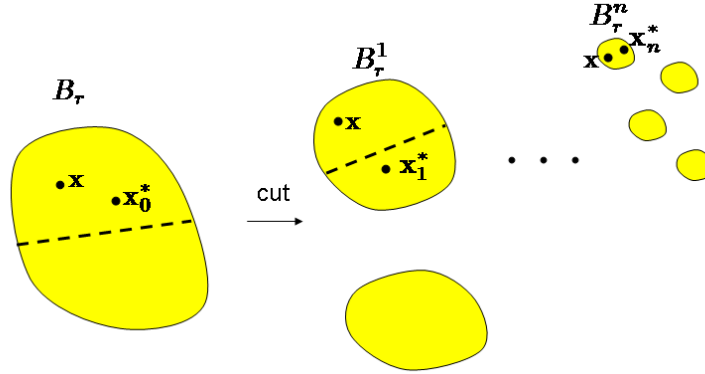
Fig. 6.7: The residually stressed body $\mathcal{B}_r$ is repeatedly cut, relieving stress. As the number of cuts $n \to \infty$, the distance between the point $\mathbf{x}$ and the stress free point $\mathbf{x}^*$ goes to zero.

a "configuration" but rather defined through the collection of tangent spaces, that is a tangent bundle. This tangent bundle is the tangent bundle of a manifold, the "virtual configuration". Reversing this map, define $\mathbf{F}_r$ as the tensor that takes $\mathcal{B}_v$ to the residually stressed body $\mathcal{B}_r$. Letting $\mathbf{F}_l$ be the deformation tensor from $\mathcal{B}_r$ to $\mathcal{B}$, i.e. the deformation that loads $\mathcal{B}_r$, define the elastic deformation tensor

$$\mathbf{A} := \mathbf{F}_l \mathbf{F}_r.$$

We now have a tensor $\mathbf{A}$ from a stress free configuration to the current configuration. However, $\mathcal{B}_v$ is not Euclidean - this is the price to pay for being unstressed - and so $\mathbf{A}$ is not in general compatible: there is no function χ_r such that $\mathbf{A} = \mathrm{grad}\chi_r$. To resolve this, the final step is to describe a deformation from the stress free reference configuration $\mathcal{B}_0$ to the stress free virtual configuration $\mathcal{B}_v$. The fundamental assumption of morphoelasticity, due to Rodriguez et al. (Rodriguez, Hoger and McCulloch, 1994), is to introduce the *growth deformation tensor* $\mathbf{G}$ which takes the tangent space of $\mathcal{B}_0$ to the tangent space of $\mathcal{B}_v$. The full deformation tensor from $\mathcal{B}_0$ to $\mathcal{B}$, denoted $\mathbf{F}$, is defined as the composition of the growth deformation $\mathbf{G}$ followed by the elastic deformation $\mathbf{A}$;

$$\mathbf{F} = \mathbf{A} \cdot \mathbf{G}. \tag{6.27}$$

Then $\mathbf{F}$ is defined from a stress free Euclidean reference configuration, and so the constitutive relation (6.24) applies. The complete theory is sketched in Figure 6.8.

Physically, the theory is as follows: $\mathbf{G}$ describes the local increase or decrease of mass in $\mathcal{B}_0$. Since $\mathbf{G}$ only informs the change of mass locally, this growth step can lead to an incompatible (but still stress free) configuration, where for instance neighboring cells overlap. Thus the growth is accompanied by an elastic response – the deformation $\mathbf{A}$ – in which material is reorganized to ensure compatibility, and which includes the
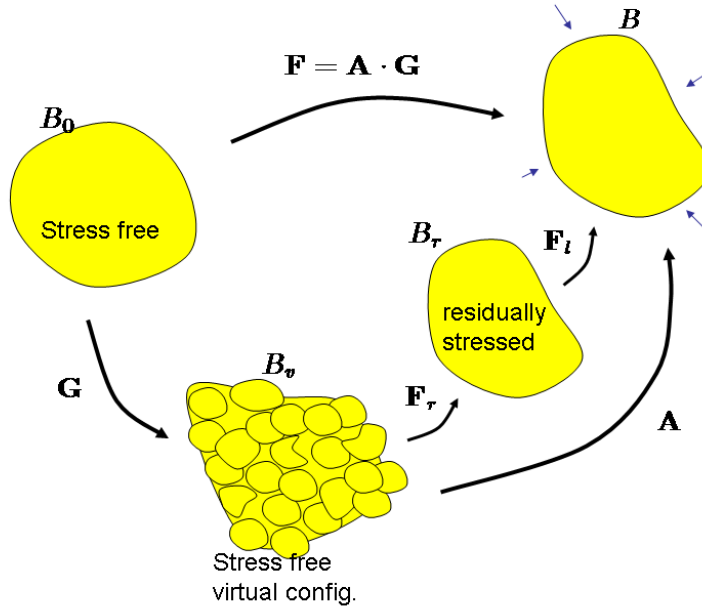
Fig. 6.8: Schematic for the decomposition of the deformation tensor $\mathbf{F}$ into the growth tensor $\mathbf{G}$ and the elastic deformation tensor $\mathbf{A}$. The tensor $\mathbf{A}$ in turn is comprised of the reorganization deformation $\mathbf{F}_r$, in which residual stress is introduced, and the deformation due to loading, $\mathbf{F}_l$.

response to any applied loads. It is through the elastic response that residual stress is introduced.

As far as practical problems and solving equations, there are two ways to view the theory. We can either define a certain growth tensor and external loads andcompute the new stressed configuration, or starting with a stressed configuration, obtain the corresponding G. In the first case, suppose the reference configuration $\mathcal{B}_0$ is known, and the growth tensor $\mathbf{G}$ is provided. The constitutive equation (6.24) transforms to

$$\mathbf{T} = \mathbf{A}\frac{\partial W}{\partial \mathbf{A}} - p\mathbf{1}. \tag{6.28}$$

In the case of mechanical equilibrium and in the absence of body forces, the governing equation is $\text{div}\mathbf{T} = \mathbf{0}$. Solving this along with appropriate boundary conditions specifying any applied loads, the deformation, as well as the stress field in the current configuration, can be completely determined. In Section 6.4 we demonstrate the process with an example.

In the second case, the question we began with was how to determine a response theory for a residually stressed material. Suppose that the current configuration $\mathcal{B}$ is known, but the reference configuration $\mathcal{B}_0$ is not. Equation (6.28) may again be

used in conjunction with $\mathrm{div}\,\mathbf{T} = 0$, in this case to solve for the deformation in the opposite direction, given that the growth deformation $\mathbf{G}$ is known. That is, if a stress free reference configuration is unknown but the form of growth which induced residual stress is known and the current configuration is also known, the reference state can be determined. Once the stress free reference state is found, the response theory is well defined. Hence, if external loading is applied to a residually stressed material, the resulting deformation can be determined in two steps: first, ignore the loading by letting $\mathbf{A} = \mathbf{F}_r$ only and solve for the "reverse deformation" to determine the stress free reference configuration $\mathcal{B}_0$. Once this is known, the external loading can be added back into the elastic deformation tensor by letting $\mathbf{A} = \mathbf{F}_l\mathbf{F}_r$, and then the deformation $\mathbf{F} = \mathbf{A} \cdot \mathbf{G}$ encompasses the induced residual stress due to growth as well as deformation due to external loading. Solving for the deformation in the forward direction enables one to determine the desired configuration with loading, $\mathcal{B}$.

6.3.1 Continuous growth

Now consider the situation where a material evolves and is residually stressed. This material can be subject to further growth. In such case, growth is applied to a residually stressed material rather than a stress-free state. This is equivalent to a subsequent growth deformation of a previously grown body, and is tied to the problem of a continuously or incrementally growing body. More broadly, this is the subject of morphodynamics. In a continuously growing body, the growth can be modeled by a relation of the form

$$\dot{\mathbf{G}} = \mathcal{G}(\mathbf{G}, \mathbf{A}, \mathbf{T}, \ldots, ; \mathbf{X}, t), \tag{6.29}$$

where the rate of growth is not necessarily constant, and might in general depend on the growth and strain tensors, stress, position, time, and potentially other factors. This can be approximated by a discrete growth process, where (6.29) is replaced by $\mathbf{G}(t+\Delta t) = \mathbf{G}(t) + \Delta t\mathcal{G}$. This relation may be seen to define an incremental growth, $\mathbf{G}_{\mathrm{inc}} := \mathbf{G}(t + \Delta t) - \mathbf{G}(t)$. At each step, the material grows according to some incremental growth law, followed by an incremental elastic response necessary for compatibility. The problem is that beyond the first step, the body is not stress free and so the constitutive relation (6.28) does not apply. One solution, given in detail in (Goriely and Amar, 2007), is for each incremental step, define a total growth tensor, defined from the initial, stress free configuration $\mathcal{B}_0$, which takes into account all previous growth increments. In this way, the ith incremental step becomes equivalent to a single step defined from a stress free state.

Consider Figure 6.9, which illustrates the idea for 2 incremental steps. The problem is that the virtual state $\mathcal{B}_{v2}$, after the second incremental step $\mathbf{G}_{\mathrm{inc}}^{(2)}$, is not stress free. The key is to unload $\mathcal{B}_{v2}$ by defining a stress free virtual configuration $\mathcal{V}_2$, and finding a deformation from $\mathcal{V}_2$ to $\mathcal{B}_{v2}$. In brief, since the growth steps do not introduce stress, the stress in $\mathcal{B}_{v2}$ arises from the elastic response $\mathbf{A}_{\mathrm{inc}}^{(1)}$ and a possible rotation of the principal axis due to $\mathbf{G}_{\mathrm{inc}}^{(2)}$. Thus, the deformation from $\mathcal{V}_2$ to $\mathcal{B}_{v2}$ is of the form $\mathbf{R}\cdot\mathbf{A}_{\mathrm{inc}}^{(1)}$, where $\mathbf{R}$ is a rotation. By inverting this map, the growth tensor $\mathbf{G}_2$ can be defined. In particular, if the growth and elastic tensors commute, as is the case with radial deformations in a spherical or cylindrical geometry, $\mathbf{G}_2$ takes the convenient form
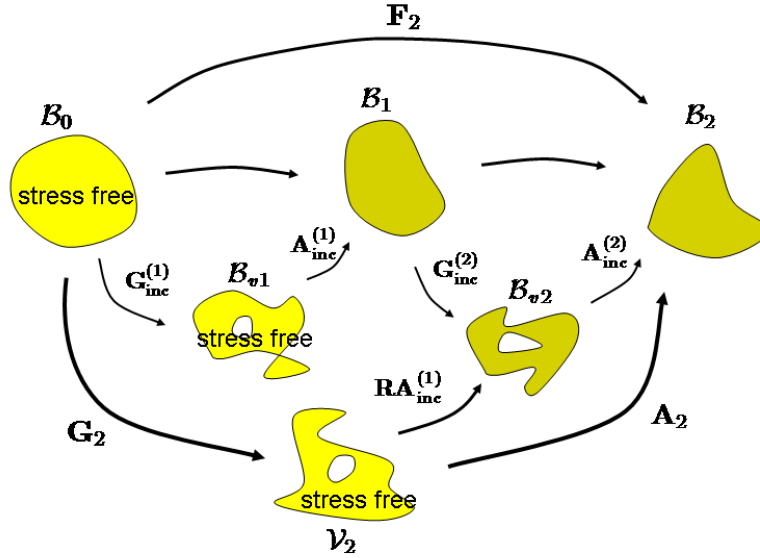
Fig. 6.9: Two incremental steps. The total growth tensor $\mathbf{G}_2$ is defined after creating a map from the stress free virtual configuration $\mathcal{V}_2$ to the stressed virtual configuration $\mathcal{B}_{v2}$.

$$\mathbf{G}_2 = \mathbf{G}_{\mathrm{inc}}^{(2)} \cdot \mathbf{G}_{\mathrm{inc}}^{(1)}. \tag{6.30}$$

Since $\mathbf{G}_2$ is defined from the stress free state $\mathcal{B}_0$, the state $\mathcal{B}_2$ can be found using the constitutive relation (6.28) for the elastic deformation $\mathbf{A}_2$ and the decomposition $\mathbf{F}_2 = \mathbf{A}_2 \cdot \mathbf{G}_2$. Extending this idea to an arbitrary number of steps is relatively straightforward, in which case the total growth becomes a product of all subsequent growth increments. In essence, the process consists of *evolving* the virtual configuration as the body grows.

6.3.2   Relationship between γ and G

To complete the basic theory of morphoelasticity, we need to relate the growth tensor $\mathbf{G}$ to the growth rate function $\gamma(\mathbf{x})$ appearing in the mass balance equation (6.19). To do this, first note that expressed in the reference frame, the mass equation is

$$\frac{d}{dt}(J\rho) = J\gamma. \tag{6.31}$$

Next, we write $J = J_A J_G$, so that J_A measures the change in volume during the elastic deformation, and J_G during the growth deformation. Let ρ_G be the density with respect to the virtual configuration, and consider an element of mass in the virtual state, $dM = \rho_G dV$. Since mass is not added in the elastic deformation, the mass in the current state is
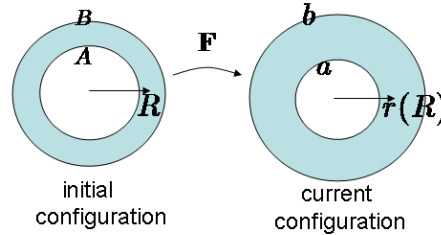
Fig. 6.10: Geometry for the growing cylindrical tube before and after deformation.

$$dm = \rho dv = dM = \rho_G dV, \tag{6.32}$$

and since $dv = J_A dV$, we have $\rho J_A = \rho_G$. Using this relationship along with the fact that $J_G = \det\mathbf{G}$, (6.31) can be written after some tensor algebra as

$$\dot{\rho} + \rho_G \mathrm{tr}(\mathbf{G}^{-1}\dot{\mathbf{G}}) = J_A\gamma. \tag{6.33}$$

In particular, if density does not change with growth, $\dot{\rho} = 0$ and the relation is

$$\rho\mathrm{tr}(\mathbf{G}^{-1}\dot{\mathbf{G}}) = \gamma. \tag{6.34}$$

In the case of constant volume $J_G = 1$, and the relationship can be written $\dot{\rho}_G = J_A\gamma$. In which case we can interpret γ as the rate of densification.

6.4 Sample problem - Growing cylindrical tube

In this section we present as a sample problem the growth and deformation of an incompressible, hyperelastic cylindrical tube. We restrict to radial deformations uniform along the tube axis. Therefore, we focus only on a cross section of the tube. The geometry is sketched in Figure 6.10. The radius of the undeformed tube is defined by $A \leq R \leq B$. The deformation is described by the function $r(R)$, which gives the location of circles of radius R after deformation. Let the inner radius in the current configuration be $r(A) = a$ and the outer radius be $r(B) = b$, so that the tube after deformation is given by $a \leq r \leq b$. This is depicted in Figure 6.10.

The deformation gradient tensor, expressed in cylindrical coordinates, is

$$\mathbf{F} = \mathrm{diag}(r'(R), r/R, 1). \tag{6.35}$$

The elastic strain tensor is $\mathbf{A} = \mathrm{diag}(\alpha_1, \alpha_2, 1)$, and the growth tensor for the symmetric growth is $\mathbf{G} = \mathrm{diag}(\gamma_1, \gamma_2, 1)$. Here γ_1 corresponds to radial growth and γ_2 to circumferential growth. Thus if $\gamma_2 = 1$ and $\gamma_1 \neq 1$, growth is purely in the radial direction, with each radial fiber increasing if $\gamma_1 > 1$ and decreasing if $\gamma_1 < 1$. The case $\gamma_1 = 1$, $\gamma_2 \neq 1$ corresponds to circumferential fibers gaining or losing mass – see Figure 6.11. If the γ_i are functions of R, the growth is inhomogeneous. Isotropic growth occurs when $\gamma_1 = \gamma_2$.
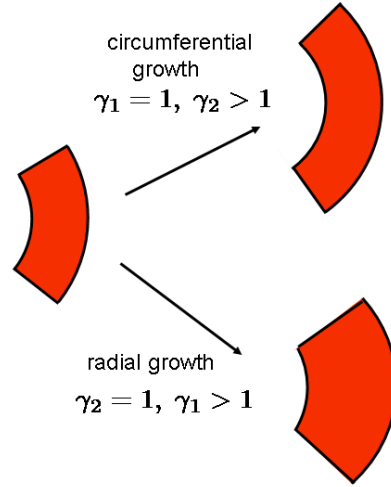
Fig. 6.11: Transformation of an area element under circumferential and radial growth.

The decomposition $\mathbf{F} = \mathbf{A} \cdot \mathbf{G}$ implies

$$r' = \alpha_1 \gamma_1, \quad r/R = \alpha_2 \gamma_2. \tag{6.36}$$

The elastic incompressibility condition is $\det \mathbf{A} = 1$. Defining $\alpha := \alpha_2$, this implies $\alpha_1 = 1/\alpha$. Together with (6.36), the deformation is thus determined by the equation $r' = \gamma_1 \gamma_2 R/r$, which after integration yields

$$r^2 - a^2 = 2 \int_A^R \gamma_1 \gamma_2 R \, dR. \tag{6.37}$$

Once the value of the inner radius a is known, the deformation is completely determined. Let t_1 and t_2 be the radial and circumferential (hoop) stresses of the Cauchy stress tensor. In the absence of body forces, the equilibrium linear momentum equation is $\mathrm{div}\mathbf{T} = \mathbf{0}$, and the only non vanishing component is

$$\frac{\partial t_1}{\partial r} + \frac{1}{r}(t_1 - t_2) = 0. \tag{6.38}$$

The constitutive equation (6.28) has components

$$t_1 = \alpha_1 W_r - p, \quad t_2 = \alpha_2 W_2 - p, \tag{6.39}$$

where $W = W(\alpha_1, \alpha_2)$ is the strain energy function and $W_i = \frac{\partial W}{\partial \alpha_i}$. Using incompressibility, we define the auxiliary function $\hat{W}(\alpha) = W(\alpha^{-1}, \alpha)$. Eliminating p in (6.39), (6.38) gives the following closed equation for the stress:

$$\frac{\partial t_1}{\partial r} = \frac{\alpha \hat{W}'(\alpha)}{r}. \tag{6.40}$$
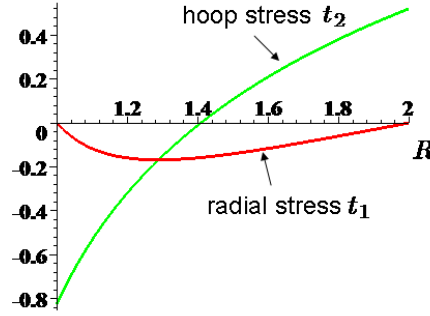
Fig. 6.12: Residual stress in a deformed neo-Hookean tube after anisotropic growth. ($A = 1$, $B = 2$, $\mu = 1$.)

Integrating (6.40) from $r = a$ to b and defining $-P = t_1(b) - t_1(a)$, we have

$$-P = \int_a^b \frac{\alpha \hat{W}'(\alpha)}{r} \, dr. \tag{6.41}$$

Note that P is the applied load on the cylindrical tube, such that $P > 0$ in the case of external pressure and $P < 0$ for internal pressure. For given growth functions γ_i and fixed pressure P, (6.41) is an equation for the unknown parameter a, since b is related to a through (6.37). Equation (6.41) may also be recast as an integral either in R or α. Once the value of a is known, the deformation is completely determined. Then the stress functions t_1 and t_2 are found via

$$t_1(r) = \int_a^r \frac{\alpha \hat{W}'(\alpha)}{r} \, dr, \quad t_2 = t_1 + \alpha \hat{W}'(\alpha). \tag{6.42}$$

As we have discussed, growth can create residual stress even in the absence of external forcing. To demonstrate, we consider a neo-Hookean material, characterized by strain energy $W = \frac{\mu}{2}(\alpha_1^2 + \alpha_2^2 - 2)$, $\mu > 0$. Set the external pressure $P = 0$, and consider anisotropic homogeneous growth, that is γ_i constant with $\gamma_1 \neq \gamma_2$. In Figure 6.12, we plot the radial and hoop stresses as functions of R for $\gamma_1 = 1.5$, $\gamma_2 = 1$. The radial growth of the tube induces a compressive residual radial stress in the material. In terms of the hoop stress, the inner radius is in compression while the outer radius is in tension. Circumferential growth, or equivalently radial resorption, creates a tensile radial stress, this is the case $\gamma_1 < 1$ (not plotted).

6.4.1   Stability

To make progress in morphoelasticity, it is usually necessary to make assumptions about the class of allowable deformations. For instance, it is typical to restrict attention to symmetric deformations, as we did in the previous example. It is reasonable to

expect an initially symmetric body subjected to symmetric growth and loading to remain symmetric, and such a restriction enables for analytical progress to be made. However, it also raises the question of stability – when can we claim that a symmetric solution of a mechanical equilibrium is stable?

The approach to stability is to add a perturbation to a finite deformation. The perturbation comes in the form of an imposed incremental deformation from a wider class of deformation, i.e. without the symmetry of the finite deformation. Let $\chi^{(0)}$ be a known finite deformation and $\mathbf{T}^{(0)}$ the related Cauchy stress, also known. We introduce an incremental deformation $\chi^{(1)}$ and consider

$$\chi = \chi^{(0)} + \epsilon \chi^{(1)}, \tag{6.43}$$

where ϵ is a small parameter characterizing the size of the imposed perturbation. The stability analysis proceeds by taking expansions of the deformation gradient and the elastic strain tensor (the growth tensor remains constant). That is, we write

$$\mathbf{F} = \mathrm{Grad}\chi = \mathbf{F}^{(0)} + \epsilon \mathbf{F}^{(1)} \mathbf{F}^{(0)}, \quad \mathbf{A} = \mathbf{A}^{(0)} + \epsilon \mathbf{A}^{(1)} \mathbf{A}^{(0)}. \tag{6.44}$$

This form of the expansions is chosen so that the incremental deformation is expressed in the current configuration. The relation $\mathbf{F} = \mathbf{A} \cdot \mathbf{G}$ implies $\mathbf{F}^{(0)} = \mathbf{A}^{(0)} \mathbf{G}$ and $\mathbf{F}^{(1)} = \mathbf{A}^{(1)}$. Incompressibility implies $\mathrm{tr}(\mathbf{F}^{(1)}) = 0$. Next, expand the Cauchy stress tensor as $\mathbf{T} = \mathbf{T}^{(0)} + \epsilon \mathbf{T}^{(1)} + O(\epsilon^2)$ and the hydrostatic pressure as $p = p^{(0)} + \epsilon p^{(1)}$. The constitutive relation (6.28) reads, to zeroth order,

$$\mathbf{T}^{(0)} = \mathbf{A}^{(0)} \cdot (W_\mathbf{A}^{(0)} - p^{(0)} \mathbf{1}) \tag{6.45}$$

and to first order

$$\mathbf{T}^{(1)} = \mathbf{A}^{(1)}(\mathbf{T}^{(0)} + p^{(0)}) + \mathcal{L} : \mathbf{A}^{(1)} - p^{(1)} \mathbf{1}, \tag{6.46}$$

where $\mathcal{L} : \mathbf{A}^{(1)} = \mathbf{A}^{(0)} W_{\mathbf{AA}}^{(0)} : \mathbf{A}^{(1)} \mathbf{A}^{(0)}$ and $W_\mathbf{A}^{(0)}$, $W_{\mathbf{AA}}^{(0)}$ are the first and second derivatives of W with respect to $\mathbf{A}$, evaluated at $\mathbf{A}^{(0)}$. The components of the 4th order tensor $\mathcal{L}$ are the instantaneous elastic moduli, these are functions of the strain components α_i and the derivatives of the strain energy $W(\alpha_1, \alpha_2)$ (see (Ogden, 1984) for a derivation and formulas).

Setting $\mathrm{div}\mathbf{T} = \mathbf{0}$ gives at $O(1)$ $\mathrm{div}(\mathbf{T}^{(0)}) = \mathbf{0}$, which is already satisfied by the finite deformation. At $O(\epsilon)$, the incremental equilibrium equation $\mathrm{div}(\mathbf{T}^{(1)}) = \mathbf{0}$ may be written (see (Ben Amar and Goriely, 2005))

$$\mathbf{A}^{(1)} \mathrm{grad} p^{(0)} + \mathrm{div}(\mathcal{L} : \mathbf{A}^{(1)}) - \mathrm{grad} p^{(1)} = 0. \tag{6.47}$$

Boundary conditions consist of prescribing either $\chi^{(1)}$ or $\mathbf{n} \cdot \mathbf{T}^{(1)}$, that is either the deformation or the traction are prescribed on the boundary.

For a given finite deformation, all terms at order zero are known. Then, Equation (6.47) plus boundary conditions form the system for the linear stability analysis. If a solution to the boundary value problem exists, a bifurcation from the set of symmetric deformations is detected. To demonstrate, in the following subsection we return to the example of a growing cylindrical tube and consider its stability.

6.4.2   Stability of the growing cylindrical tube

As before, we consider only a cross section of the cylindrical tube, and so all quantities are assumed independent of z; instability is characterized by a bifurcation to a solution with angular dependence. At a point of instability, the tube "buckles" circumferentially to a non-circular cross section. This is the phenomenon of mucosal folding, which is seen to occur in a number of biological tissues, including airways (Lambert, Codd, Alley and Pack, 1994), esophagus (Liao, Zhao, Yang and Gregersen, 2007), and the gastrointestinal tract (Gregersen, 2003).

Working in polar coordinates, the general form for the incremental deformation is

$$\chi^{(1)} = [u(r,\theta), v(r,\theta)]. \tag{6.48}$$

Then

$$\mathbf{F}^{(1)} = \mathbf{A}^{(1)} = \begin{bmatrix} u_r & \dfrac{u_\theta - v}{r} \\[2mm] v_r & \dfrac{u + v_\theta}{r} \end{bmatrix}. \tag{6.49}$$

The incompressibility condition is

$$\operatorname{tr}(\mathbf{A}^{(1)}) = u_r + (u + v_\theta)/r = 0. \tag{6.50}$$

Inserting (6.49) into the incremental equilibrium equation (6.47) yields two differential equations involving the unknown functions u, v, and $p^{(1)}$. To simplify, these functions are assumed to be of the form

$$
\begin{aligned}
u(r,\theta) &= f(r)\sin(n\theta) \\
v(r,\theta) &= g(r)\cos(n\theta) \\
p^{(1)}(r,\theta) &= h(r)\sin(n\theta).
\end{aligned} \tag{6.51}
$$

In this manner instability is examined as a function of buckling mode n, the number of folds in the buckled state. Making these substitutions and using the incompressibility condition (6.50) to solve for $g(r)$ in terms of $f(r)$, the system can be simplified to a single 4th order differential equation for $f(r)$:

$$B_4 f^{(4)}(r) + B_3 f'''(r) + B_2 f''(r) + B_1 f'(r) + B_0 f(r) = 0 \tag{6.52}$$

where

$$B_4 = \gamma_2^2 R^2, \quad B_3 = \frac{2\gamma_2\left(\gamma_1\,\gamma_2\,R^2 + 2\,r^2\right)}{r\gamma_1},$$

$$B_2 = -\frac{3\,\gamma_2{}^4 R^4 \gamma_1 - 8\,\gamma_2{}^3 R^2 r^2 + n^2 \gamma_2{}^4 R^4 \gamma_1 + r^4 n^2 \gamma_1}{r^2 \gamma_2{}^2 R^2 \gamma_1},$$

$$B_1 = \frac{-3\,r^4 n^2 \gamma_1\,\gamma_2\,R^2 + 2\,r^6 n^2 + 3\,\gamma_2{}^5 R^6 \gamma_1 - 4\,\gamma_2{}^4 R^4 r^2 + n^2 \gamma_2{}^5 R^6 \gamma_1 - 2\,\gamma_2{}^4 R^4 n^2 r^2}{r^3 \gamma_1\,\gamma_2{}^3 R^4},$$

$$B_0 = \frac{-3\,\gamma_2{}^4 R^4 \gamma_1 + 4\,\gamma_2{}^3 R^2 r^2 + 3\,n^2 \gamma_2{}^4 R^4 \gamma_1 - 4\,\gamma_2{}^3 R^2 n^2 r^2 - r^4 n^2 \gamma_1 + r^4 n^4 \gamma_1}{r^4 \gamma_2{}^2 R^2 \gamma_1}.$$

Note that $R = R(r)$ is determined from the finite deformation via (6.37). The system is closed with the addition of boundary conditions – a typical choice is to set both shear and normal stresses equal to zero on the inner and outer edges. Alternatively, part of the tube may be held fixed in which case one might require the perturbation to vanish at one boundary.

With given boundary conditions, a solution to the boundary value problem indicates the onset of instability. The approach is to leave one parameter in the system free and search for a critical value of that parameter at which the BVP has a solution. This parameter might be the strain α at the inner boundary $r = a$, the applied pressure P, or the rate of growth. Here we briefly consider the case of homogeneous anistropic growth: we set $\gamma_2 = 1$, $\gamma_1 \neq 1$, and let the external pressure P be the bifurcation parameter.

For thin shells, an asymptotic analysis may be used to solve the ODE (6.52) directly, but in the general case numerical methods are needed. One approach is to assume

$$f(r) = a_1 \xi_1(r) + a_2 \xi_2(r). \tag{6.53}$$

We then treat the system as an initial value problem at $r = a$, and solve two copies of the system with linearly independent initial conditions for the $\xi_i(r)$, such that both ξ_1 and ξ_2 satisfy the two boundary conditions at $r = a$ (note that 8 total initial conditions must be given: $\xi_i(a), \xi_i'(a), \xi_i''(a), \xi_i'''(a), \ \ i = 1, 2$, whereas the boundary conditions at $r = a$ only provide 4 total conditions). Integrate both ξ_i forward to $r = b$ and form the determinant of boundary values at $r = b$

$$D(P) = \begin{vmatrix} c_1(\xi_1(b); P) & c_1(\xi_2(b); P) \\ \\ c_2(\xi_1(b);)) & c_2(\xi_2(b); P) \end{vmatrix} \tag{6.54}$$

where $c_1(f(b)) = 0$, $c_2(f(b)) = 0$ are the two boundary conditions at $r = b$. Due to linearity, if the determinant $D(P) = 0$, then there exist values a_1 and a_2 for which $f(r)$ given by (6.53) solves the BVP. The parameter P is iterated on until the determinant vanishes.

In Figure 6.13, the critical value of P is plotted as a function of the radial growth γ_1, for various modes n and for fixed tube thickness $A/B = 0.5$. For a given γ_1, the smallest value of P_{cr} over all modes is the critical external pressure, denoted P_{cr}^*, at which the tube becomes unstable and buckles, and the corresponding mode n_{cr} is the critical buckling mode. For $P < P_{cr}^*$, the tube remains circular.

The value of P_{cr}^* for $\gamma_1 = 1$ is the critical pressure without growth, which occurs for mode $n_{cr} = 2$. Growth can have both stabilizing and destabilizing effects. For this thickness, radial resorption ($\gamma_1 < 1$) is destabilizing – P_{cr}^* is smaller for all values of $\gamma_1 < 1$ than for $\gamma_1 = 1$. On the other hand, radial growth ($\gamma_1 > 1$) has a stabilizing effect initially but is destabilizing for large γ_1. Observe that the "strongest" tube possible, i.e. highest value of P_{cr}^*, is for $\gamma_1 \approx 1.72$, the point marked b in Figure 6.13. With this in mind, one could imagine as an application using differential growth and bifurcation curves such as those in Figure 6.13 in the *design* of biomaterials in which the fine tuning of critical pressure is desired.
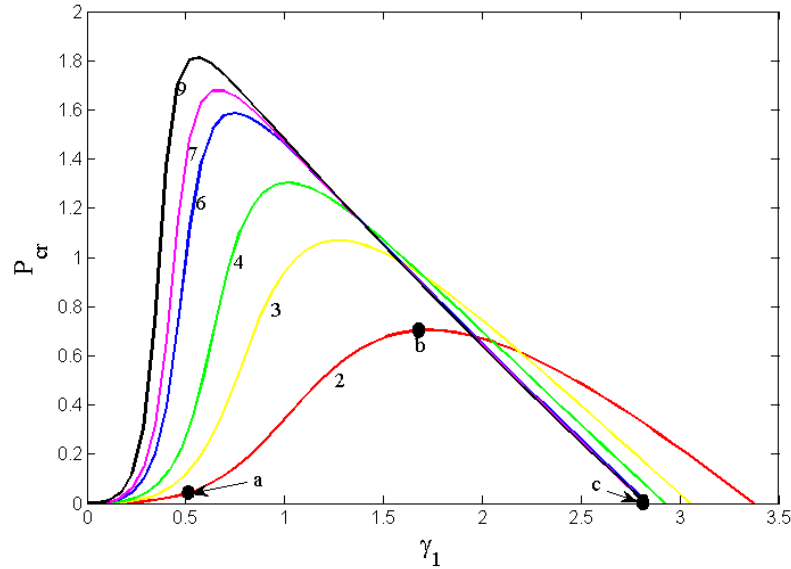
Fig. 6.13: Critical buckling pressure as a function of radial growth γ_1 for modes $n = 2, 3, 4, 6, 7, 9$.

Along with the change in critical pressure, growth has a dramatic effect on the shape after bifurcation as well. Samples of the tube after deformation are provided in Figure 6.14 for the three points marked a, b, c in Figure 6.13. Note that point c corresponds to buckling due solely to differential growth (since $P_{cr}^* = 0$). It should not be surprising that differential growth can cause instability without any external loading, since it was demonstrated earlier that growth can induce residual stress. However, growth also changes the geometry of the body which can have a stabilizing effect (as stubbier body are less likely to suffer buckling than thin ones). Further results can be established regarding the effect of tube thickness on the buckling. Also, a greater range of possibilities opens up when considering a tube with two connected layers of material with different stiffness properties. For a detailed exploration, see (Moulton and Goriely, 2010).

6.5 Conclusions

Growth processes are abundant in nature, yet the rules governing them are inherently complex. Moreover, growth occurs in different forms, and the varying types of growth may be fundamentally different both on a biological and a mathematical level. Theoretical and experimental analyses to understand growth processes on a mechanical level have been developed over a number of years and a good deal of understanding has been achieved. Of particular interest is the residual stress that can be induced in
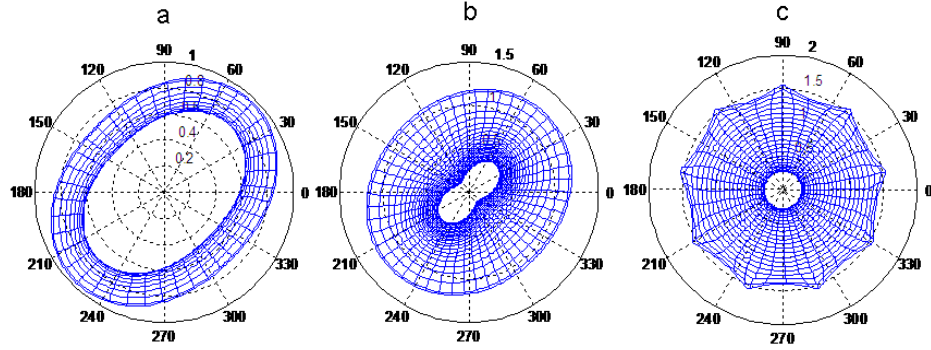
Fig. 6.14: Tube deformation after bifurcation corresponding to the points marked in Figure 6.13. Note that the amplitude has been chosen to illustrate the structure - it has no physical significance since the shapes are produced in a linear stability analysis.

a body by growth – exactly how and why it arises, and the role that it plays in elastic materials are key issues.

We have presented in this chapter a summary of some of the key ideas and challenges underlying morphoelasticity. We illustrated the basic concept by considering one-dimensional growth. Even in one dimension, interesting and challenging questions arise. We then developed the full three-dimensional theory, with the question of how to think about a residually stressed body culminating in the fundamental idea of morphoelasticity, the decomposition of the deformation tensor into growth and elastic deformations. We then illustrated the process by considering a growing cylindrical tube, demonstrating how growth can alter material properties through a stability analysis.

Many problems remain unsolved in morphoelasticity, and the theory itself is far from complete. One of the big future challenges in the field is in morphodynamics. In particular, we discussed a continuously growing body as satisfying a relation

$$\dot{\mathbf{G}} = \mathcal{G}(\mathbf{G}, \mathbf{A}, \mathbf{T}, \dots, ; \mathbf{X}, t); \tag{6.55}$$

however finding appropriate biologically derived growth laws $\mathcal{G}$ presents an important challenge. Moreover, the incremental growth theory presented here largely becomes impractical when studying non-symmetric deformations or geometries, and stability in a continuous framework presents a much greater difficulty.

References

Ben Amar, M. and Goriely, A. (2005). Growth and instability in elastic tissues. *J. Mech. Phys. Solids*, **53**, 2284–2319.

Cowin, S. C. (2004). Tissue growth and remodeling. *Annu. Rev. Biomed. Eng.*, **6**, 77–107.

Détienne, P. and Thiel, J. (1988). Monograph on wapas in french guiana. *Bois et Forêts des Tropiques*, **216**, 43–68.

Gooday, G. W. and Trinci, A. P. J. (1980). *Wall structure and biosynthesis in fungi. In: Gooday, G.W., Lloyd, D., Trinci, A.P.J. (Eds.), The Eukaryote Microbial Cell.* Cambridge University Press, London.

Goriely, A. and Amar, M. Ben (2007). On the definition and modeling of incremental, cumulative, and continuous growth laws in morphoelasticity. *Biomech Model in Mechanobiol*, **6**(5), 289–296.

Goriely, A. and Ben Amar, M. (2005). Differential growth and instability in elastic shells. *Phys. Rev. Lett.*, **94**(19), 198103.

Goriely, A. and Tabor, M. (2003). Biomechanical models of hyphal growth in actinomycetes. *J. Theor Biol*, **222**, 211–218.

Gregersen, H. (2003). *Biomechanics of the Gastrointestinal Tract: New Perspectives in Motility Research and Diagnostics.* Springer, London.

Howard, R. J. and Valent, B. (1996). Breaking and entering: host penetration by the fungal rice blast pathogen, *Magnaporthe grisea. Annu. Rev. Microbiol.*, **50**, 491–512.

Humphrey, J. D. (2003). Continuum biomechanics of soft biological tissues. *Proc. Roy. Soc. Lond. A*, **459**, 3–46.

Lambert, R. K., Codd, S. L., Alley, M. R., and Pack, R. J. (1994). Physical determinants of bronchial mucosal folding. *J. Appl. Physiology*, **77, No. 3**, 1206–1216.

Liao, D., Zhao, J., Yang, J., and Gregersen, H. (2007). The oesophageal zero-stress state and mucosal folding from a giome perspective. *World J. Gastroenterol*, **13(9)**, 1347–1351.

Moulton, D. E. and Goriely, A. (2010). Buckling instability of a growing cylindrical tube. preprint.

Ogden, R. W. (1984). *Non-linear Elastic Deformation.* Dover, New York.

Ogden, R. W. (2003). *Nonlinear Elasticity with Application to Material Modelling.* Warsaw: Polish Academy of Sciences, Institute of Fundamental Technological Research.

Rodriguez, E. K., Hoger, A., and McCulloch, A. (1994). Stress-dependent finite growth in soft elastic tissues. *J. Biomech*, **27**, 455–467.

Skalak, R. and Hoger, A. (1997). Kinematics of surface growth. *J. Math. Biol.*, **35**, 869–907.

Taber, L. A. and Humphrey, J. D. (2001). Stress-modulated growth, residual stress,

and vascular heterogeneity. *J. Biomech. Eng.*, **123**, 528–535.

Thompson, D. W. (1992). *On Growth and Form: The Complete Revised Edition.* Dover, New York.

Vandiver, R. and Goriely, A. (2008). Tissue tension and axial growth of cylindrical structures in plants and elastic tissues. *Europhys. Lett. (EPL)*, **84**(58004).