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The nonlinear dynamics of elastic tubes conveying a fluid

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ABSTRACT

The Kirchhoff equations for elastic tubes are modified to include the effect of fluid flow. Using the techniques of linear and nonlinear analysis specially developed for the Kirchhoff equations, the effect of the fluid flow on the basic twist-to-writhe instability is investigated. The results suggest an intriguing modification of the bifurcation threshold due to the flow. Beyond threshold the buckled tube acquires a slight curvature which modifies the flow rate and results in a correction to nonlinearity of the amplitude equation governing the deformation dynamics.

1. Introduction

The Kirchhoff equations for elastic rods provide a fundamental model to study the statics and dynamics of slender elastic bodies subject to a wide variety of boundary conditions and body forces. In his landmark work (see Dill (1992)) Kirchhoff assumed that the cross-section of the rod is much smaller than its length thereby justifying a cross-sectional averaging of the rod stresses and, by assuming the curvature of the filament is small relative to its length, he was able to derive on the basis of asymptotic arguments linear constitutive relations between the moment and the rod curvatures. The net result is the celebrated Kirchhoff equations: a system of six coupled, nonlinear PDEs of second-order in time and arclength – the latter variable providing the intrinsic coordinate of the center-line of the rod. A more contemporary approach (see Antman (2005)) casts Kirchhoff's assumptions about the deformation in terms of the theory of Global Constraints and the assumption of hyper-elasticity allows for more general constitutive relations. The static Kirchhoff equations, a system of six coupled nonlinear ODEs, have been studied extensively both analytically and numerically (Love, 1892; Antman and Kenney, 1981; Mielke and Holmes, 1988; Li and Maddocks, 1996; Champneys et al., 1997; McMillen and Goriely, 2002). By contrast, the time-dependent equations are far less tractable and until recently defied analytical treatment (Antman and Liu, 1979; Maddocks and Dichmann, 1994; Coleman et al., 1995a; Antman, 1996; Beliaev and Il'ichev, 1996; Goriely and Tabor, 1996; Goriely and Tabor, 2000; Durickovic et al., 2009). Similarly, their numerical solution is difficult and developing stable, energy conserving, algorithms has proved to be a challenging problem (Dichmann et al., 1996; Weiss, 2002).

Studies of the Kirchhoff equations, be they static, time-dependent, or purely numerical have been extensive and wide ranging. They have been applied to the study of problems as diverse as the super-coiling of DNA strands (Benham, 1979, 1983; Tanaka and Takahashi, 1985; Benham, 1989; Hunt and Hearst, 1991; Coleman et al., 1995b; Manning et al., 1996; Westcott et al., 1997; Tobias et al., 2000; Thompson et al., 2002; Hoffman et al., 2003; Hoffman, 2004) and the buckling of ocean cables (Zajac, 1962; Coyne, 1990; Costello, 1990; Neukirch and van der Heijden, 2002) – both of which, despite their enormous disparities in scale and mechanical properties, exhibit the same fundamental twist-to-writhe instability, namely the looping that occurs when a critical twist threshold is reached.

The study of pipe flow also represents another venerable chapter in the history of continuum mechanics. The effect of bends on fluid flow was first discussed by Thomson in his studies of winding rivers (Thomson, 1876; Thomson, 1877). This and other subsequent studies (Grindley and Gibson, 1908; Eustice, 1910) led to the work of Dean (1928) who gave the first analytical treatment of curvature effects on pipe flow. Series expansions in terms of what is now termed the Dean number provided a description of the secondary flows and showed how curvature reduced the mean flow rate. Later work by Germano (1982, 1989) on flow in helical pipes examined the effects of both pipe curvature and torsion on the flow and showed that for pipes of circular cross-section torsion makes a lower order modification to the flow than the curvature. While the above mentioned work was primarily concerned with

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rigid pipes there has been a body of research devoted to flow in flexible and rigid pipes (Berger et al., 1983; Zabielski and Mestel, 1998; Gammack and Hydon, 2001; Roberts, 2004) with applications ranging from the instabilities exhibited by pipes (Lundgren et al., 1979; Gregory and Paidoussis, 1966a; Gregory and Paidoussis, 1966b) to blood flow in arteries (Caro et al., 1996; Pedley, 1980).

In this paper we address the following simple but basic question: how is the twist-to-writhe instability exhibited by an elastic tube affected by a fluid flow down the tube? To tackle this problem we combine the Kirchhoff model of elastic filaments with a Dean-like representations of the flow, and use the nonlinear analysis techniques developed in earlier papers (Goriely et al., 2001) to study the resulting instability.

The paper is organized in the following way. We first derive, subject to a number of simplifying assumptions, the Kirchhoff equations for an elastic tube conveying a fluid. The linear and nonlinear analysis techniques developed by two of us are then used to describe the twist-to-writhe instability and derive the nonlinear amplitude equations that capture the post-bifurcation dynamics. These results are then used to study the delay of bifurcation that arises for a tube of finite length, a result that has particular significance with respect to possible experimental studies. The model is then extended to include the effects of tube curvature on the mean flow rate and we show how this leads to a modification of the amplitude equations.

2. The Kirchhoff equations with a fluid flow

To establish notation and basic principles, we begin by briefly summarizing the derivation of the Kirchhoff equations in the fluid-free case. Let $\mathbf{r} = \mathbf{r}(s,t) = r_x\mathbf{e}_x + r_y\mathbf{e}_y + r_z\mathbf{e}_z$ be a space curve in the Euclidean frame $\{\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z\}$, parameterized by the arc length s and time t, representing the center-line of the rod (or tube) which is assumed to be inextensible and unshearable. For each value of s and t a local orthonormal coordinate system, the director basis $\{\mathbf{d}_1,\mathbf{d}_2,\mathbf{d}_3\}$, is associated with the curve by identifying the vector $\mathbf{d}_3(s,t)$ as the tangent vector of $\mathbf{r}$ at s and taking the vector pair $\{\mathbf{d}_1,\mathbf{d}_2\}$ to span the plane normal to $\mathbf{d}_3$ such that $\{\mathbf{d}_1,\mathbf{d}_2,\mathbf{d}_3\}$ forms a right-handed triad; *i.e.*

$$\mathbf{d}_3(s,t) = \mathbf{r}'(s,t), \quad \mathbf{d}_1 \times \mathbf{d}_2 = \mathbf{d}_3, \quad \mathbf{d}_2 \times \mathbf{d}_3 = \mathbf{d}_1 \tag{1}$$

where $()'$ denotes differentiation with respect to s. By taking the vectors $\{\mathbf{d}_1,\mathbf{d}_2\}$ to coincide with the principle axes of the rod cross-section the rotation of the basis captures the actual twist of the rod. The spatial geometry of the rod can be reconstructed at all times by integration of the tangent vector, *i.e.* $\mathbf{r}(s,t) = \int_0^s \mathbf{d}_3(s,t)\,ds$.

The kinematics of the rod is determined by the evolution of the director basis. From the orthonormality of this basis it follows that:

$$\mathbf{d}_i' = \sum_{j=1}^{3} K_{ij}\mathbf{d}_j \quad i = 1,2,3, \tag{2}$$

$$\dot{\mathbf{d}}_i = \sum_{j=1}^{3} W_{ij}\mathbf{d}_j \quad i = 1,2,3, \tag{3}$$

where $(\dot{\ })$ stands for the time derivative. W and K are the antisymmetric 3 by 3 matrices:

$$K = \begin{pmatrix} 0 & \kappa_3 & -\kappa_2 \\ -\kappa_3 & 0 & \kappa_1 \\ \kappa_2 & -\kappa_1 & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}. \tag{4}$$

The elements of K make up the components of the twist vector, $\boldsymbol{\kappa} = (\kappa_1,\kappa_2,\kappa_3)$, and the elements of W define the components of the spin vector, $\boldsymbol{\omega} = (\omega_1,\omega_2,\omega_3)$. The two linear systems (2) and (3) must be compatible: thus by cross-differentiation it follows that:

$$W' - \dot{K} = [W,K] \equiv W \cdot K - K \cdot W. \tag{5}$$

We first recall the dynamics of an inextensible and unshearable rod (or tube) of circular cross-section. The force and moment are averaged over each normal cross-section along the center-line axis and resolved in the director basis so that the resultant force and moment are expressed locally as

$$\mathbf{F} = \sum_{i=1}^{3} f_i\mathbf{d}_i, \quad \mathbf{M} = \sum_{i=1}^{3} m_i\mathbf{d}_i$$

In the absence of external body forces and couples, conservation of linear and angular momentum give the well-known equations of motion (Coleman et al., 1993):

$$\mathbf{F}' = \rho A\ddot{\mathbf{r}}, \tag{6}$$

$$\mathbf{M}' + \mathbf{d}_3 \times \mathbf{F} = \rho I(\mathbf{d}_1 \times \ddot{\mathbf{d}}_1 + \mathbf{d}_2 \times \ddot{\mathbf{d}}_2), \tag{7}$$

where ρ is the density, A the cross-sectional (material) area and I is the cross-section's moment of inertia (or second moment of area). For a solid rod of radius a, $I = \pi a^4/4$, and for a circular tube with outer radius a and inner radius b,

$$I = \pi(a^4 - b^4)/4 \tag{8}$$

Differentiation of (6) with respect to arc-length and use of (1) enables the equations to be expressed entirely in terms of the director basis. The system is closed by choosing an appropriate constitutive relationship between the moment and the strains. The standard linear relationship takes the form:

$$\mathbf{M} = EI\left[(\kappa_1 - \kappa_1^{(u)})\mathbf{d}_1 + (\kappa_2 - \kappa_2^{(u)})\mathbf{d}_2\right] + \mu J(\kappa_3 - \kappa_3^{(u)})\mathbf{d}_3, \tag{9}$$

where E is the Young modulus, μ the shear modulus, and $J = 2I$ for a rod of circular cross-section. The $\kappa_i^{(u)}$ are the intrinsic curvatures of the filament which we will now set to zero corresponding to the case of a naturally straight and untwisted rod.

We now introduce fluid flow into the model and assume that it is a fully developed pressure-driven flow that does not transmit torque to the tube itself. At first, we will consider the tube to be infinitely long and perform a stability analysis around that state. Then, following the methodology developed in (Goriely et al., 2001), the effect of finite length and boundary conditions will be captured by a weakly nonlinear analysis close to the bifurcation.

Following the work of (Steindl and Troger (1995), Steindl and Troger (1996)), we represent the motion of the fluid as a cross-sectionally averaged mean flow with a given constant velocity U_0 along the tangent vector. Later we will discuss the case of nonconstant flow in which the flow rate is modified by the curvature of the tube. Assuming the absence of external damping and external loads on the tube motion, the balance of linear momentum of the fluid conveying tube is given by

$$\mathbf{F}' = m_T\mathbf{a}_T + m_F\mathbf{a}_F - (m_T + m_F)g\mathbf{e}_z \tag{10}$$

where the terms on the right hand side represent, respectively, tube inertia, fluid inertia, and the effect of gravity; with m_T and m_F denoting the tube and fluid masses per unit length. The first term is familiar from the earlier discussion, namely

$$m_T\mathbf{a}_T = \rho_T A_T\ddot{\mathbf{r}} \tag{11}$$

where ρ_T is the tube material density, $A_T = \pi(a^2 - b^2)$ is the cross-section area of the tube wall, and it is understood that $\dot{\mathbf{r}} \equiv \mathbf{r}_t$ denotes derivative with respect to time at fixed arc-length (*i.e.* the partial t derivative). The second term is determined by the following consideration. Assuming that the position of a material point of fluid along the tube is determined by $\mathbf{r} = \mathbf{r}(s(t), t)$ we note that the total t derivative is

$$\frac{d\mathbf{r}}{dt} = \frac{\partial \mathbf{r}}{\partial s}\frac{ds}{dt} + \frac{\partial \mathbf{r}}{\partial t} = U_0\mathbf{d}_3 + \dot{\mathbf{r}} \tag{12}$$

where $U_0 = ds/dt$ is the constant, cross-sectionally averaged, fluid speed down the tube. Thus the fluid inertial term is

$$\begin{aligned} m_F\mathbf{a}_F &= m_F\frac{d^2\mathbf{r}}{dt^2} = m_F\left(U_0^2\mathbf{d}_3' + 2U_0\dot{\mathbf{d}}_3 + \ddot{\mathbf{r}}\right) \\ &= \rho_F A_F\left(U_0^2\mathbf{d}_3' + 2U_0\dot{\mathbf{d}}_3 + \ddot{\mathbf{r}}\right) \end{aligned} \tag{13}$$

where ρ_F is the fluid density and $A_F = \pi b^2$ is the inner cross-sectional area of the tube. The form of the first two terms on the right hand side are suggestive of the following physical interpretations:

$$m_F U_0^2\mathbf{d}_3' \sim \text{centrifugal force}, \quad 2U_0 m_F\dot{\mathbf{d}}_3 \sim \text{Coriolis force}$$

However, we note that the terms "centrifugal" and "Coriolis" used here are not associated with an actual rotational effect but simply reflect dimensional and mathematical similarities with the well-know rotational effects. Differentiating both sides of (10) with respect to s gives

$$\mathbf{F}'' = (\rho_T A_T + \rho_F A_F)\ddot{\mathbf{d}}_3 + \rho_F A_F\left(U_0^2\mathbf{d}_3'' + 2U_0\dot{\mathbf{d}}_3'\right) \tag{14}$$

where we have now dropped the gravitational term.

Working on the assumption that Newtonian fluids do not transmit torque, we use the standard Kirchhoff form for the conservation of angular momentum and write

$$\mathbf{M}' + \mathbf{d}_3 \times \mathbf{F} = \rho_T I(\mathbf{d}_1 \times \ddot{\mathbf{d}}_1 + \mathbf{d}_2 \times \ddot{\mathbf{d}}_2) \tag{15}$$

where I is given by (8). Eqs. (14) and (15) represent the simplest possible generalization of the Kirchhoff equations (6) and (7) to include the effect of a fluid flow. These equations can be used to study the effect of the fluid on the twist-to-writhe instability. As will be shown below, the centrifugal term has a direct impact on the dispersion relation characterizing the bifurcation, and the Coriolis term on the form of the ensuing amplitude equations for post-bifurcation behavior.

The presence of the fluid flow introduces additional physical parameters into the system and hence a variety of possible scalings of the equations. Here we set the characteristic time and length scales according to:

$$\tau = \left(\frac{I\rho_F}{EA_F}\right)^{1/2}, \quad l = \left(\frac{I}{A_F}\right)^{1/2}$$

In the case of a finite length rod, one could also use the length of the rod as the characteristic length scale. In this case an additional dimensionless parameter would appear in front of the rotary inertia terms (the r.h.s. of (15)). This scaling is typically the best choice for numerical analysis. However, for computing the associated amplitude equations, we choose the tube diameter as the typical length scale thereby keeping the rotary inertia terms in the analysis. The variables in (14), (15) equations are

$$\begin{aligned} & t \to t\sqrt{I\rho_F/A_F E}, \quad s \to s\sqrt{I/A_F}, \\ & \mathbf{F} \to \mathbf{F}A_F E, \quad \mathbf{M} \to \mathbf{M}E\sqrt{A_F I}, \\ & \boldsymbol{\kappa} \to \boldsymbol{\kappa}\sqrt{A_F/I}, \quad \boldsymbol{\omega} \to \boldsymbol{\omega} \end{aligned} \tag{16}$$

resulting in

$$\mathbf{F}'' = (1+\alpha)\ddot{\mathbf{d}}_3 + \delta^2\mathbf{d}_3'' + 2\delta\dot{\mathbf{d}}_3', \tag{17}$$

$$\mathbf{M}' + \mathbf{d}_3 \times \mathbf{F} = \mathbf{d}_1 \times \ddot{\mathbf{d}}_1 + \mathbf{d}_2 \times \ddot{\mathbf{d}}_2, \tag{18}$$

$$\mathbf{M} = \kappa_1\mathbf{d}_1 + \kappa_2\mathbf{d}_2 + \Gamma\kappa_3\mathbf{d}_3. \tag{19}$$

where $\Gamma = \mu J/EI$. We will refer to the above system as the Kirchhoff-fluid equations. While the angular momentum equation and constitutive relations are the same as before, the linear momentum equation sees the appearance of two new dimensionless parameters:

$$\alpha = \frac{\rho_T A_T}{\rho_F A_F} \sim \frac{\text{tube mass/unit length}}{\text{fluid mass/unit length}} \tag{20}$$

and

$$\delta = \left(\frac{\rho_F U_0^2}{E}\right)^{1/2} \sim \left(\frac{\text{fluid stress}}{\text{elastic stress}}\right)^{1/2} \tag{21}$$

In the case of a thin-walled tube $\alpha \ll 1$ and in what follows we will make this approximation. It is of interest to write out (17) in component form:

$$\begin{aligned} & f_1'' - f_1(\kappa_2^2 + \kappa_3^2) + f_2(\kappa_1\kappa_2 - \kappa_3') + (f_3 - \delta^2)(\kappa_2' + \kappa_1\kappa_3) \\ & \quad - 2\kappa_3 f_2' + 2\kappa_2 f_3' - 2\delta(\omega_2' + \omega_1\kappa_3) = (1+\alpha)(\dot{\omega}_2 + \omega_1\omega_3) \end{aligned} \tag{22}$$

$$\begin{aligned} & f_2'' - f_2(\kappa_1^2 + \kappa_3^2) + f_1(\kappa_3' + \kappa_1\kappa_2) + (f_3 - \delta^2)(\kappa_2\kappa_3 - \kappa_1') \\ & - 2\kappa_1 f_3' + 2\kappa_3 f_1' - 2\delta(-\omega_1' + \omega_2\kappa_3) = (1+\alpha)(-\dot{\omega}_1 + \omega_2\omega_3) \end{aligned} \tag{23}$$

$$\begin{aligned} & f_3'' - (f_3 - \delta^2)(\kappa_1^2 + \kappa_2^2) + f_1(\kappa_1\kappa_3 - \kappa_2') + f_2(\kappa_2\kappa_3 + \kappa_1') \\ & - 2\kappa_2 f_1' + 2\kappa_1 f_2' - 2\delta(\omega_1\kappa_1 + \omega_2\kappa_2) = -(1+\alpha)(\omega_1^2 + \omega_2^2) \end{aligned} \tag{24}$$

which reveals that the centrifugal term has resulted in a renormalization of the tension, $f_3 \to f_3 - \delta^2$ and that the Coriolis term has introduced the spin-twist couplings, $\omega_i\kappa_j$.

By inserting the constitutive relationship for $\mathbf{M}$ (19) in (18) one obtains explicit differential relationships between the lateral forces and the strains. For an inextensible rod the tangential component of the force, *i.e.* f_3, corresponds to the tension (or compression if negative) in the filament. Together with the twist and spin equations (2) and (3), one obtains, overall, a system of 9 equations for 9 unknowns $(\mathbf{f}, \boldsymbol{\kappa}, \boldsymbol{\omega})$ which we write in the shorthand form:

$$\mathbf{E}(\mathbf{f}, \boldsymbol{\kappa}, \boldsymbol{\omega}; s, t) = 0. \tag{25}$$

3. Linear and nonlinear analysis

In a series of papers (Goriely and Tabor, 1997a; Goriely and Tabor, 1997b; Goriely et al., 2001; Goriely and Tabor, 2000) we developed an intrinsic, *i.e.* director-based, perturbation scheme that enables one to analyze the linear stability of elastic rods subject to perturbations, such as the injection of twist into the rod, and showed how to use this scheme to develop nonlinear amplitude equations to describe the post-bifurcation dynamics. The details are given in the cited papers and here we just sketch out the key features to lay the ground-work for the analysis of the fluid flow problem. We start with a stationary solution $\boldsymbol{\mu}^{(0)} = (\mathbf{f}^{(0)}, \boldsymbol{\kappa}^{(0)})$ so that $\mathbf{E}(\mathbf{f}^{(0)}, \boldsymbol{\kappa}^{(0)}, \mathbf{0}; s, t) = 0$ which is associated with a stationary basis $\{\mathbf{d}_1^{(0)}, \mathbf{d}_2^{(0)}, \mathbf{d}_3^{(0)}\}$. The key step is the development of a perturbation scheme in which the director basis associated with the deformed configuration of the filament is expanded around the basis of an unperturbed stationary solution $\boldsymbol{\mu}^{(0)} = (\boldsymbol{\kappa}^{(0)}, \mathbf{f}^{(0)})$, namely:

$$\mathbf{d}_i = \mathbf{d}_i^{(0)} + \epsilon\mathbf{d}_i^{(1)} + \epsilon^2\mathbf{d}_i^{(2)} + \ldots \quad i = 1, 2, 3, \tag{26}$$

Application of the orthonormality condition $\mathbf{d}_i.\mathbf{d}_j = \delta_{ij}$ at each order leads to an expression for the perturbed basis in terms of the unperturbed basis. In particular we can show that

$$\mathbf{d}_i^{(1)} = \sum_{j=1}^{3} A_{ij}^{(1)}\mathbf{d}_j^{(0)} \tag{27}$$

where $A^{(1)}$ is the *antisymmetric* matrix:

$$A^{(1)} = \begin{pmatrix} 0 & \alpha_3^{(1)} & -\alpha_2^{(1)} \\ -\alpha_3^{(1)} & 0 & \alpha_1^{(1)} \\ \alpha_2^{(1)} & -\alpha_1^{(1)} & 0 \end{pmatrix}, \quad (28)$$

The expansion (26) is then used to expand the force vector and the twist and spin matrices, and on substitution into the Kirchhoff equations leads to the dynamical variational equations which determine the (linear) stability of the stationary solutions with respect to time-dependent modes. The variational equations can be written as

$$\mathbf{L_E}(\boldsymbol{\mu}^{(0)}; s, t).\boldsymbol{\mu}^{(1)} = 0, \quad (29)$$

where $\mathbf{L_E}$ is a second-order differential operator in s and t whose coefficients depend on s through the stationary solution $\boldsymbol{\mu}^{(0)}$. The components of the 6-dimensional vector $\boldsymbol{\mu}^{(1)} = (\mathbf{f}^{(1)}, \boldsymbol{\alpha}^{(1)})$ are $\mathbf{f}^{(1)}$, the first order correction to the forces and $\boldsymbol{\alpha}^{(1)}$, the components of the vector encapsulating the first order correction to the director basis.

The linear system (29) can be used to determine stability of a given static configuration (characterized by the ($\boldsymbol{\mu}^{(0)}$)) by setting the components of $\boldsymbol{\mu}^{(1)}$ to be of the form:

$$\boldsymbol{\mu}^{(1)} = e^{\sigma t}\left(A\xi e^{ins} + A^*\xi^* e^{-ins}\right) \quad (30)$$

where $()^*$ stands for the complex conjugate, n is the mode number which is continuous for an infinite rod and replaced by $k\pi/L$ for a rod of finite length L. The amplitude, A, is arbitrary at the level of the linear theory and can only be determined by a nonlinear analysis. Substitution of (30) into the variational equation (29) gives the dispersion relations $\Pi(\sigma, n) = \det(\mathbf{L_E}) = 0$ which relates the growth rates σ to the mode number n. A mode, n, is deemed unstable when $\Re(\sigma(n))$ becomes positive, and the corresponding stability threshold is identified by the neutral curve(s) $\sigma(n) = 0$ determined by solving $\Pi(0, n) = 0$. To date we have only studied the case where $\sigma \in \Re$. Indeed, the dispersion relation can have other solutions corresponding to vibration modes. However, these modes are not spontaneously unstable and their amplitude is at most of the size of the perturbation itself. For basic stability consideration these modes do not need to be considered, but could play a subtle role in the resulting nonlinear behavior.

Here we are concerned with the case of a straight tube subjected to applied twist and tension. For an infinite tube lying along the x-axis the static solutions to the Kirchhoff fluid equations are easily shown to be:

$$\boldsymbol{\kappa}^{(0)} = (0, 0, \gamma) \quad \mathbf{f}^{(0)} = \left(0, 0, P^2\right). \quad (31)$$

where γ is the twist density in the rod and P^2 is the (prescribed) tension along the filament. Here, we use the notation P^2 for comparison with previous papers and to emphasize the fact that the stress applied to the filament is tensile.

In the particular case of the static solution (31), the general computation leading to the linearized system (29) and its solution is lengthy but straightforward and leads to the dispersion relation

$$\Pi(\sigma, n) = \Pi_1(n)\Pi_2(\sigma, n)\Pi_3(\sigma, n; \gamma, Q) \quad (32)$$

where

$$\Pi_1(n) = n^2, \quad (33)$$

$$\Pi_2(\sigma, n) = n^2\Gamma + 2\sigma^2, \quad (34)$$

and $\Pi_3(\sigma, n; \gamma, Q)$ is a high order polynomial in n depending on the parameters γ and $Q = \sqrt{P^2 - \delta^2}$.

The dispersion branch $\Pi_1(n) = 0$ corresponds to a *tension mode* with eigenvector

$$\mu^{(1)} = \xi_1 = (0, 0, 1, 0, 0, 0) \quad (35)$$

and the branch $\Pi_2(\sigma, n) = 0$ corresponds to a *twist mode* with eigenvector

$$\mu^{(1)} = \xi_2 = (0, 0, 0, 0, 0, 1). \quad (36)$$

The neutral curve associated with the third dispersion branch is determined from:

$$\Pi_3(0, n; \gamma, Q) = (\gamma^2 - n^2)^2\left[\left(\gamma^2(\Gamma - 1) - Q^2 - n^2\right)^2 - \gamma^2(\Gamma - 2)^2 n^2\right] = 0, \quad (37)$$

This branch of the dispersion relation gives a parabolic neutral curve corresponding to a linearly unstable helical mode with critical values

$$\gamma_c = \frac{2Q}{\Gamma}, \quad n_c = \frac{(2 - \Gamma)Q}{\Gamma} \quad (38)$$

The associated eigenvector is now

$$\mu^{(1)} = \xi_3 = (-Q^2, -iQ^2, 0, -i, 1, 0). \quad (39)$$

These results parallel those found for the fluid free case (Goriely et al., 2001) and comparison of the critical values (38) for these two cases indicates that the fluid flow lowers the bifurcation threshold. However, these results have been obtained for an infinite tube and we will later show how the threshold is modified for a tube of finite length and given boundary conditions.

Although the linear analysis can identify the initial instabilities as a function of the system parameters it is only valid for very short times and cannot describe the post-bifurcation dynamics. The techniques of weakly nonlinear analysis enable one to develop an asymptotic expansion of the solution amplitudes, as a function of longer space and time scales, close to the bifurcation threshold (Newell, 1974). In this regime, the distance from the bifurcation point is of the order of the perturbation itself. The new scales, on which the arbitrary linear amplitudes vary, can be deduced from the dispersion relations.

For our case of a straight twisted rod or tube we take the twist, γ, as the control parameter of the system and set

$$\gamma = \gamma_c + \Psi\epsilon^2 \quad (40)$$

where Ψ is an order one constant used for book-keeping purposes. The choice of ϵ^2 (rather than ϵ) is for the notational convenience of avoiding certain square roots that would otherwise appear and this enables us to write all the required expansions in integer powers of ϵ. In order to determine the new space and time scales we set

$$n = n_c + \epsilon^\alpha \Delta n, \quad \sigma = \epsilon^\beta \Delta\sigma$$

and expanding $\Pi_3(\sigma, n; \gamma, Q)$ about the critical point (n_c, γ_c) gives the lowest balance between the terms

$$Q^2(\Delta n)^2\epsilon^{2\alpha} - 2iQ\delta(\Delta\sigma)\epsilon^\beta - Q^3\Gamma\epsilon^2\Psi$$

Balance is now achieved if

$$\alpha = 1, \quad \beta = 2$$

which tells us that the stretched variables scale as

$$S = \epsilon s, \quad T = \epsilon^2 t \quad (41)$$

We note that these scalings are quite different from the fluid-free case where one may show (Goriely and Tabor, 1997b) that $T = \epsilon t$, $S = \epsilon s$. This difference can be traced back to the presence of the cross-derivative term $\dot{\mathbf{d}}_3'$ introduced by the fluid flow. The expansion of the dispersion relations suggests that the linear part of the amplitude equation, for amplitude $A(S,T)$, will look like

$$-2i\delta Q\frac{\partial A}{\partial T}-Q^2\frac{\partial^2 A}{\partial S^2}-Q^3\Gamma\Psi A$$

which follows from equating Δn and $\Delta\sigma$ to S and T derivatives respectively.

Taking into account the expansion of all the variables in the bifurcation parameter and the new scales, we now look for solutions of the full system (25) order by order in ϵ:

$$0(\epsilon^0):\quad \mathbf{E}(\boldsymbol{\mu}^{(0)};s,t)=0 \tag{42}$$

$$0(\epsilon^1):\quad \mathbf{L_E}(\boldsymbol{\mu}^{(0)};s,t).\boldsymbol{\mu}^{(1)}=0 \tag{43}$$

$$0(\epsilon^2):\quad \mathbf{L_E}(\boldsymbol{\mu}^{(0)};s,t).\boldsymbol{\mu}^{(2)}=\mathbf{H}_2(\boldsymbol{\mu}^{(1)}) \tag{44}$$

$$0(\epsilon^3):\quad \mathbf{L_E}(\boldsymbol{\mu}^{(0)};s,t).\boldsymbol{\mu}^{(3)}=\mathbf{H}_3(\boldsymbol{\mu}^{(1)},\boldsymbol{\mu}^{(2)}) \tag{45}$$

$$\vdots$$

where the $\mathbf{H}_i$ are certain functions of their given arguments and their derivatives (with respect to S and T), and the vector $\boldsymbol{\mu}^{(0)}=(\boldsymbol{\alpha}^{(0)},\mathbf{f}^{(0)})$ corresponds to the static solution.

To order $0(\epsilon)$, the (linear) solution is given by a superposition of the neutral modes; namely

$$\boldsymbol{\mu}^{(1)}=C(S,T)\xi_1+B(S,T)\xi_2+A(S,T)\xi_3e^{ins}+A^*(S,T)\xi_3^*e^{-ins} \tag{46}$$

where $C(S,T)$, $B(S,T)$, and $A(S,T)$ represent, respectively, the slowly varying amplitudes of the tension mode, the axial twist mode and the unstable helical mode.

In order to derive an equation governing the evolution of these amplitudes on the new scales we consider the higher-order equations in (42)–(45) and look for solubility conditions determined by the Fredholm Alternative. At $O(\epsilon^2)$ we find the solubility condition $C(S,T)=0$. Since we want to include the tension mode in the calculation we add it to the particular solution satisfying (44), *i.e.* we set $\boldsymbol{\mu}^{(2)}\to\boldsymbol{\mu}^{(2)}+C(S,T)\xi_1$, where we note that this added term satisfies the associated homogeneous problem. Thus we see that the tension mode enters the amplitude equation analysis at $O(\epsilon^2)$. The Fredholm alternative conditions at $O(\epsilon^3)$ give the amplitude equations satisfied by $B(S,T)$ and $A(S,T)$ and at $O(\epsilon^4)$ the equation for $C(S,T)$. The net result is the following system of equations:

$$-\frac{2i\delta}{Q}\frac{\partial A}{\partial T}-\frac{\partial^2 A}{\partial S^2}=A\left(2Q\Gamma\Psi+Q\Gamma\frac{\partial B}{\partial S}-C-2Q^2\Gamma|A|^2\right), \tag{47}$$

$$\frac{\partial^2 B}{\partial S^2}=2Q\frac{\partial}{\partial S}|A|^2, \tag{48}$$

$$\frac{\partial^2 C}{\partial S^2}=-2Q^2\frac{\partial^2}{\partial S^2}|A|^2. \tag{49}$$

The scalings of the stretched variables (41) results in quite different linear operators in these equation compared to those that were found in the fluid free case (Goriely et al., 2001). In particular, the amplitude equation for A involves a first time derivative, and the twist density, B, and tension amplitude, C are completely slaved to the mode amplitude $|A|^2$. Integration of (48) and (49) and substitution into (47) gives the nonlinear amplitude equation

$$-\frac{2i\delta}{Q}\frac{\partial A}{\partial T}-\frac{\partial^2 A}{\partial S^2}=A\left(2Q\Gamma\Psi+2Q^2|A|^2\right), \tag{50}$$

for the helical amplitude. If the tension mode had been neglected the system would reduce to a simple linear wave equation.

4. Delay of bifurcation

We now consider a tube of length $2\pi L$ where $-\pi L\leqslant s\leqslant\pi L$. The length factor 2π and the symmetric disposition of the tube being chosen for computational convenience. The effect of the finite length on the bifurcation threshold can be determined by studying solutions to the static system

$$\frac{\partial^2 A}{\partial S^2}=-A\left(2Q\Gamma\Psi+Q\Gamma\frac{\partial B}{\partial S}-C-2Q^2\Gamma|A|^2\right), \tag{51}$$

$$\frac{\partial^2 B}{\partial S^2}=2Q\frac{\partial}{\partial S}|A|^2, \tag{52}$$

$$\frac{\partial^2 C}{\partial S^2}=-2Q^2\frac{\partial^2}{\partial S^2}|A|^2, \tag{53}$$

subject to the condition that the ends are free to slide but that their tangent vectors are fixed. Thus we require the various amplitudes to satisfy the boundary conditions (with respect to the scaled length $S=\epsilon s$)

$$A(-\pi L\epsilon)=A(\pi L\epsilon)=0;\quad B(-\pi L\epsilon)=B(\pi L\epsilon)=0;$$
$$C(-\pi L\epsilon)=C(\pi L\epsilon)=0.$$

Integration of the twist and tension mode equations gives

$$\frac{\partial B}{\partial S}=2Q\frac{\partial}{\partial S}|A|^2+2K_3, \tag{54}$$

and

$$C=-2Q^2|A|^2+K_4S+K_5. \tag{55}$$

The boundary conditions on C immediately give $K_4=K_5=0$. Substitution of (54) and (55) into (51) gives

$$\frac{\partial^2 A}{\partial S^2}=-A\left(2Q\Gamma K_1+2Q^2\Gamma|A|^2\right), \tag{56}$$

where

$$K_1=K_3+1 \tag{57}$$

and we have set $\Psi=1$. Substituting $A=re^{i\phi}$ in (56) gives

$$r''-\frac{l}{r^3}+\lambda r+\mu r^3=0, \tag{58}$$

where prime denotes differentiation with respect to S, and $l=r^2\phi'$, $\lambda=2Q\Gamma K_1$, $\mu=2Q^2$. For the zero angular momentum case, $l=0$, the equation for $r(s)$ can be integrated in terms of the Jacobi elliptic *cn* function to give

$$r=\alpha cn(\beta S|k), \tag{59}$$

where

$$\alpha=\sqrt{\frac{2\lambda k^2}{\mu(1-k^2)}},\quad \beta=\sqrt{\frac{\lambda}{(1-2k^2)}},\quad k^2\leqslant 1/2. \tag{60}$$

The boundary condition $r(-\pi L\epsilon)=r(\pi L\epsilon)=0$ gives

$$\pi L\epsilon\beta=\pi L\epsilon\sqrt{\frac{\lambda}{(1-2k^2)}}=K(k),$$

where $K(k)$ is the standard complete elliptic integral of the first kind. Thus

$$K_1=\frac{(1-2k^2)K^2(k)}{\pi^2\epsilon^2L^2Q\Gamma}. \tag{61}$$

Integration of (54) gives

$$B(S)=\frac{2Q|\alpha|^2}{k^2}\frac{E(S\beta)}{\beta}+2\left(-\frac{Q|\alpha|^2}{k^2}+Q|\alpha|^2+K_3\right)S+K_5, \tag{62}$$

where E is the elliptic integral of the second kind and we have used the standard result $\int_0^u cn^2(u|k)\,du=(E(u)-(1-k^2)u)/k^2$. Applying the boundary condition $B(\pi\epsilon L)=0$ gives $K_5=0$ and hence

$$|\alpha|^2=\frac{1}{Q}\frac{k^2}{\left(\frac{E(k)}{K(k)}-1+k^2\right)}(1-K_1). \tag{63}$$

Since $|\alpha|^2 \geqslant 0$ it follows that $1 - K_1 \geqslant 0$ and on using (61) and (40) we obtain

$$\gamma \geqslant \gamma_c + \frac{(1-k^2)K^2(k)}{Q\Gamma\pi^2 L^2}. \tag{64}$$

Thus we see that the bifurcation threshold is delayed by an amount inversely proportional to the square of the tube length. Noting that $K(0) = \pi/2$ we see the limiting form of (64) is

$$\gamma \geqslant \gamma_c + \frac{1}{4Q\Gamma L^2}. \tag{65}$$

If Q is replaced by P this is the same result obtained for the fluid-free case in which the tension mode is neglected and the corresponding stationary amplitude equation for A is linear (Goriely and Tabor, 1997b).

5. Effect of tube curvature beyond threshold

At the bifurcation threshold the tube changes from a straight to a helical configuration and we investigate the effect that this appearance of curvature might have on the flow and, hence, the post-bifurcation dynamics. Our model of the modified fluid flow draws on Dean's classic work (Dean, 1927, 1928). In analyzing the effect of curvature he made the fundamental geometric assumption that the ratio of the pipe's cross-sectional radius, r, is small compared to the pipe's circularly coiled radius R, *i.e.* $d = r/R \ll 1$, in other words the curvature of the pipe is small. We can view Dean's assumption as analogous to Kirchhoff's assumption that $a/L \ll 1$. After an appropriate rescaling of the fluid velocity components Dean derived dimensionless fluid equations (in a toroidal coordinate system) depending on d and the Dean number

$$K = 2dRe^2$$

where Re is the Reynolds number associated with the flow. We note that the Dean number is directly proportional to the pipe curvature $1/R$. Dean developed series expansions for the fluid flow about a straight pipe flow (*i.e.* $d = 0$) in powers of K and was able to find expressions for the secondary flow structure. A cross-sectional average gives a well-known formula for the nondimensional volume flux, Q, of the form

$$Q = \pi a^2 \overline{W}\left(1 - 0.0306(K/576)^2 + 0.0120(K/576)^4 + \cdots\right) \tag{66}$$

where $\overline{W}$ is the mean velocity in the straight pipe about which the flow is perturbed. This formula shows that the curvature induced corrections to the mean flow are proportional to powers of the square of the pipe curvature. It also shows that the corrections are small: for $K \sim 576$, the flux is reduced by about 2%. A clear discussion of Dean's work (and a careful discussion of the various definitions of the Dean number, of which the K used above is one, that are scattered in the literature) can be found in the review article by Berger et al. (1983).

The above results suggest that a valid strategy for our analysis is to represent the mean velocity in the form

$$U = U_0(1 - \tilde{a}\kappa^2 + \tilde{b}\kappa^4), \tag{67}$$

where we only keep the correction up to order 4 in the curvature $\kappa = \sqrt{\kappa_1^2 + \kappa_2^2}$, and the expansion coefficients $\tilde{a}$ and $\tilde{b}$, which have the dimensions L^2 and L^4 respectively, can be deduced from (66). Eq. (67) is, in effect, an expansion of the fluid flux in powers of the Dean number for a fixed Reynolds number and tube radius. As mentioned in the introduction, the work of Germano (1982, 1989) showed that for helical pipes of circular cross-section torsion makes a lower order correction to the flow and, accordingly, we do not include any torsion dependence in (67).

The fluid inertial term (13) is modified by the curvature corrections to U to give

$$m_F\mathbf{a}_F - \ddot{\mathbf{r}} = 2U_0(\kappa_1\dot{\kappa}_1 + \kappa_2\dot{\kappa}_2)(-\tilde{a} + 2\tilde{b}\kappa^2)\mathbf{d}_3 + U_0^2(1 - \tilde{a}\kappa^2 + \tilde{b}\kappa^4)^2\mathbf{d}_3' + 2U_0(1 - \tilde{a}\kappa^2 + \tilde{b}\kappa^4)\dot{\mathbf{d}}_3 \tag{68}$$

The Kirchhoff fluid equations (14) and (15) are modified accordingly and now take the form

$$\mathbf{F}'' = (1+\alpha)\ddot{\mathbf{d}}_3 + \delta^2(1 - a\kappa^2 + b\kappa^4)^2\mathbf{d}_3'' + 2\delta\Big((\kappa_1\dot{\kappa}_1 + \kappa_2\dot{\kappa}_2)(-a + 2b\kappa^2)\mathbf{d}_3' + (1 - a\kappa^2 + b\kappa^4)\dot{\mathbf{d}}_3'\Big), \tag{69}$$

$$\mathbf{M}' + \mathbf{d}_3 \times \mathbf{F} = \mathbf{d}_1 \times \ddot{\mathbf{d}}_1 + \mathbf{d}_2 \times \ddot{\mathbf{d}}_2, \tag{70}$$

$$\mathbf{M} = \kappa_1\mathbf{d}_1 + \kappa_2\mathbf{d}_2 + \Gamma\kappa_3\mathbf{d}_3. \tag{71}$$

where the parameters α and δ are as before and a, b are the dimensionless coefficients $\tilde{a}/l^2$, $\tilde{b}/l^4$.

Our perturbative analysis, both linear and nonlinear, can now be applied to this extended system. When expanding the curvatures we observe that

$$\kappa^2 = \kappa_1^2 + \kappa_2^2 = \left(\left(\kappa_1^{(0)}\right)^2 + \left(\kappa_2^{(0)}\right)^2\right) + 2\epsilon\left(\kappa_1^{(0)}\kappa_1^{(1)} + \kappa_2^{(0)}\kappa_2^{(1)}\right) + \epsilon^2\left(\left(\kappa_1^{(1)}\right)^2 + \left(\kappa_2^{(1)}\right)^2\right) + O(\epsilon^3) \tag{72}$$

and, hence, for the case of an initially straight (and twisted) rod for which $\kappa_1^{(0)} = \kappa_2^{(0)} = 0$, the curvature corrections will not enter until $O(\epsilon^2)$. This means that the variational equations, the associated null vectors, and the scalings are the same as before. Carrying out the multiple-scales expansion we obtain the hierarchy

$$0(\epsilon^1): \quad \mathbf{L_E}(\boldsymbol{\mu}^{(0)}; s, t).\boldsymbol{\mu}^{(1)} = 0 \tag{73}$$

$$0(\epsilon^2): \quad \mathbf{L_E}(\boldsymbol{\mu}^{(0)}; s, t).\tilde{\boldsymbol{\mu}}^{(2)} = \tilde{\mathbf{H}}_2(\boldsymbol{\mu}^{(1)}) \tag{74}$$

$$0(\epsilon^3): \quad \mathbf{L_E}(\boldsymbol{\mu}^{(0)}; s, t).\tilde{\boldsymbol{\mu}}^{(3)} = \tilde{\mathbf{H}}_3(\boldsymbol{\mu}^{(1)}, \tilde{\boldsymbol{\mu}}^{(2)}) \tag{75}$$

$$\vdots$$

where $\boldsymbol{\mu}^{(0)}$, $\boldsymbol{\mu}^{(1)}$ and the linear operator $\mathbf{L_E}(\boldsymbol{\mu}^{(0)}; s, t)$ are the same as before, while $\tilde{\boldsymbol{\mu}}^{(2)}, \tilde{\boldsymbol{\mu}}^{(3)}, \ldots$ and the functions $\tilde{\mathbf{H}}_2, \tilde{\mathbf{H}}_3, \ldots$ are significantly complicated by the curvature corrections to the flow. Despite this added complexity, the system is tractable and leads to the modified system of amplitude equations:

$$-\frac{2i\delta}{Q}\frac{\partial A}{\partial T} - \frac{\partial^2 A}{\partial S^2} = A\left(2Q\Gamma\Psi + Q\Gamma\frac{\partial B}{\partial S} - C - 2Q^2(\Gamma + 4\delta^2 a)|A|^2\right) \tag{76}$$

$$\frac{\partial^2 B}{\partial S^2} = 2Q\frac{\partial}{\partial S}|A|^2 \tag{77}$$

$$\frac{\partial^2 C}{\partial S^2} = -2Q^2(1 + 2\delta^2 a)\frac{\partial^2}{\partial S^2}|A|^2 \tag{78}$$

Thus we see, at the level of the nonlinear analysis, that the effect of the post-bifurcation curvature on the flow is a (small) correction to the nonlinearity in the equations governing the helical amplitude and tension modes. That only the first correction term in the flow (67) appears in the result should not be too surprising in view of the curvature expansion (72). As before the equations for B and C can be integrated leading to the single amplitude equation

$$-\frac{2i\delta}{Q}\frac{\partial A}{\partial T} - \frac{\partial^2 A}{\partial S^2} = A\left(2Q\Gamma\Psi + 2Q^2(1 - 2\delta^2 a)|A|^2\right) \tag{79}$$

6. Discussion

Using our model of the Kirchhoff equations for an elastic tube conveying a fluid we are able to show, on the basis of a linear stability analysis specifically developed for Kirchhoff rods, that threshold for the twist-to-writhe conversion of a straight, infinite, twisted tube is lowered by the presence of a flow. A nonlinear analysis of the post-bifurcation dynamics gives nonlinear equations for the unstable amplitude from which we are able to determine the delay of bifurcation for a tube of finite length. The delay formula (65) reveals a subtle interplay between the tube length and the flow. Recalling (38), an expansion of (65) in powers of δ gives

$$\gamma = \gamma_1 + \frac{1}{\Gamma P}\left(\frac{1}{8L^2P^2} - 1\right)\delta^2 + O(\delta^4) \tag{80}$$

where

$$\gamma_1 = \frac{2P}{\Gamma} + \frac{1}{4P\Gamma L^2}$$

corresponds exactly to the delay in the absence of a flow. The second term on the right-hand side of (80) can change sign depending on the tube length L. It is easy to deduce the critical length

$$L_c = \frac{1}{2\sqrt{2}P}$$

below which the delay is increased and above which the delay is reduced. This suggests, in principle, a way of experimentally studying the bifurcation phenomenon. In the absence of a flow one can set, for a given tube length, $L > L_c$, and applied tensile load P^2, a value of the twist slightly above the critical value γ_1. From (80) one can then deduce the value of δ, and hence the (mean) flow speed that would lower the threshold to the critical value needed to observe the twist-writhe instability. For illustrative purposes, consider the following example. For a thin walled rubber tube with internal diameter $d_i = 1.0$ cm and outer diameter $d_o = 1.1$ cm, $A_F = \pi(d_i/2)^2 \approx 8 \times 10^{-5}$ m^2 and $l = (I/A_F)^{1/2} \approx 1.7 \times 10^{-3}$ m. We take a typical rubber elastic modulus of $E = 10$ MPa and assume incompressibility, *i.e.* $\Gamma = 2/3$. For a tensile load of $f_3 = 1$ N, the dimensional tension is $P^2 = f_3/A_F E \approx 0.00125$ and hence $P \approx 0.035$. For this value of P the critical length is $L_c \approx 10.1$ (in dimensionless units). If we choose $L > L_c = 11.0$, we can estimate the fluid-free, delayed critical twist $\gamma_1 \approx 0.282$. With this choice of parameters, the flow correction factor $\frac{1}{\Gamma P}(\frac{1}{8L^2P^2} - 1)\delta^2 \approx -6.72\delta^2$. Imagine setting γ_1 to 5% above its critical value, *i.e.* increasing it by approximately *0.014* twist units. We can now estimate the flow that will negate this increment has the associated value of $\delta^2 \approx 0.0021$. This corresponds to a flow velocity of $U = \sqrt{\delta^2 E/\rho_f} \approx 4.6$ m/s. For a fluid such as water this corresponds to a rather high Reynolds number ($Re \sim 5.1 \times 10^4$). Estimates for different parameter values and configurations may suggest experimentally feasible regimes.

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