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Anticavitation and differential growth in elastic shells

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Abstract Elastic anticavitation is the phenomenon of a void in an elastic solid collapsing on itself. Under the action of mechanical loading alone typical materials do not admit anticavitation. We study the possibility of anticavitation as a consequence of an imposed differential growth. Working in the geometry of a spherical shell, we seek radial growth functions which cause the shell to deform to a solid sphere. It is shown, surprisingly, that most material models do not admit full anticavitation, even when infinite growth or resorption is imposed at the inner surface of the shell. However, void collapse can occur in a limiting sense when radial and circumferential growth are properly balanced. Growth functions which diverge or vanish at a point arise naturally in a cumulative growth process.

Keywords elasticity · Differential growth · anticavitation · cavitation

1 Introduction

The effect of growth, swelling, or more generally residual stress, on the property of elastic materials is particularly important for the modeling of biological materials. Indeed, growth and remodelling are the hallmark of mechanical biology [47]. Typically, volumetric growth, the change of mass or density of a volume element, is anisotropic and inhomogeneous. Therefore, local changes of volume are usually incompatible in the sense that the local deformation tensor due to growth alone is not the gradient of a mapping [46]. The consequence of this incompatibility of growth is the presence of residual stress in the material, that is stresses that are present even in the absence of external loads [27], a common occurrence in many biological tissues such as arteries, tumour spheroids, and plants [8, 10, 12, 16, 28, 30, 50, 51, 7, 40, 52, 13, 54]. The modeling of growing elastic tissues with residual stresses in nonlinear elasticity can be done through the multiplicative decomposition of the deformation

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tensor [11,31,33,42,26], based on elasto-plasticity [9,24,35,53], into a growth part characterizing the local change of a volume element and an elastic part ensuring the integrity of the body and giving rise to residual stresses. This theory, while still lacking a rigorous foundation [17], has been successfully applied to many physiological systems such as arteries [49,2,23,22], muscles [48], tumor [4] and plants [52]. The theoretical analysis of residual stresses in growing elastic bodies has revealed that growth-induced stresses can trigger both mechanical instability [20] and, in the case of elastic membranes, elastic cavitation [34].

Elastic cavitation is the phenomenon in which a cavity or void appears in the interior of an elastic body. This problem has received considerable interest in the literature, starting with the pioneering experimental work of Gent and Lindley [18] and the seminal analysis of Ball [5]. Ball considered solid elastic spheres under uniform hydrostatic loads and found the critical pressure at which a branch of radially symmetric configurations with an internal cavity bifurcates from the undeformed configuration. Since then, cavitation has received considerable attention and has been studied for various situations – compressibility versus incompressibility, anisotropy, and composite layers have all been analyzed in different combinations for various material models ([15,41,29,25,44,45,39,14,32], among others). In these studies, cavitation occurs as a consequence of external mechanical loading. More recently, several authors [37,38,34], have shown that growth or swelling can also induce cavitation and cavitation has been suggested as a mechanism for stem hollowing in plants [21].

Given far less consideration, and the subject of the present paper, is the “reverse problem” of anticavitation or void collapse. That is, under what conditions would a pre-existing cavity close on itself. The collapse of a cavity is a common occurrence in fluids [6]. In nonlinear elasticity, however, it has scarcely been studied. One analysis was conducted by Abeyaratne and Hou [1]. Following Ball’s analysis of the critical pressure on a solid sphere to induce cavity formation, Abeyaratne and Hou explored the critical pressure on spherical shells needed to collapse the inner void. In doing so, they found a necessary and sufficient condition on the strain-energy function of the material such that void collapse can occur under sufficient loading. Most commonly used models of elastic materials do not satisfy the condition. This is surprising in that for these materials, anticavitation through external pressure alone is impossible even though cavitation may be possible. It seems to be easier to open a cavity in a sphere than close a cavity in a shell.

In this paper, we study a *growing* elastic body containing a cavity or void in its reference unstressed configuration and explore the possibility of collapse and disappearance of the void as a function of the material and form and rate of growth. In particular, we consider whether growth can alter the material behaviour as to allow for collapse. In particular, as discussed previously, growth can induce residual stress. Residual stress can have both stabilizing and destabilizing effects [3]. Similarly, we show here that for the anticavitation problem, the development of residual stress serves as a primary force in opposition to the full collapse of the void.

For simplicity and to allow for analytical progress, we consider here a spherically symmetric shell, and ask whether growth may be imposed upon the shell such that it deforms to a solid sphere. These ideas can be easily generalised to cylindrical shells. We begin with the formulation of the problem through nonlinear elasticity, adding the component of growth via the formulation of multiplicative decomposition [43], in which the deformation tensor is given by the product of a growth tensor and an elastic strain tensor. The specific form of growth is captured by two functions (of radius) describing the amount of material that is added or removed in the radial and circumferential directions. We explore whether there exist functions that admit anticavitation. This is largely dependent on the strain energy function for the material. For a large class of materials, we show that anticavitation is in fact not possible for most physically relevant growth functions, but can be achieved in the proper limits when singular and/or vanishing growth functions are allowed. This analysis is shown to be relevant in the context of cumulative growth.

2 Setup

2.1 Background

We consider the symmetric growth and deformation of a spherical shell composed of an incompressible, hyperelastic material. The general question we explore is whether symmetric growth, i.e. addition of material, may occur such that the shell deforms to a solid sphere. Let the inner and outer radii in the reference configuration be given by $R = A$ and $R = B$, respectively, and the deformed shell be described in the current configuration by the function $r(R)$, which gives the radius of a sphere with initial radius R. Let $r(A) = a$ and $r(B) = b$, so that after deformation the shell in the current configuration has boundary radii a and b. For this map, the geometric deformation tensor is given by $\mathbf{F} = \mathrm{diag}(r', r/R, r/R)$ [36], where primes denote differentiation with respect to R. The elastic strain tensor is $\mathbf{A} = \mathrm{diag}(\alpha_1, \alpha_2, \alpha_2)$, where the index 1 denotes the radial direction and index 2 the circumferential directions. The elastic incompressibility condition is $\det(\mathbf{A}) = 1$, from which $\alpha_1 = \alpha^{-2}$ (where we have denoted $\alpha := \alpha_2$)

The symmetric growth is described by two functions, $\gamma_1(R)$ and $\gamma_2(R)$, which describe the addition (or resorption) of material in the radial and circumferential directions, respectively. In the case of homogeneous growth, with each γ_i a constant: if $\gamma_2 = 1$, material is added in the radial direction if $\gamma_1 > 1$, whereas material is lost if $\gamma_1 < 1$. If $\gamma_1 = 1$, growth is circumferential for $\gamma_2 > 1$ while $\gamma_2 < 1$ corresponds to circumferential resorption. Generally, it is the ratio γ_1/γ_2 that dictates the form of growth, so that radial growth and circumferential growth are qualitatively similar up to an isotropic growth. Here we allow these to be functions of radius.

So $\gamma_1 = \gamma_2 = 1 \Rightarrow$ no growth?

The growth tensor is $\mathbf{G} = \mathrm{diag}(\gamma_1, \gamma_2, \gamma_2)$. Using the multiplicative decomposition $\mathbf{F} = \mathbf{AG}$, we obtain $r' = \gamma_1/\alpha^2$, and $r/R = \alpha\gamma_2$. From these, the deformation may be written as

$$r^3 - a^3 = 3\int_A^R \gamma_1(\tilde{R})\gamma_2^2(\tilde{R})\tilde{R}^2\, d\tilde{R}. \tag{1}$$

Note also the relation

$$\alpha = \frac{r}{\gamma_2 R} = \frac{\left(a^3 + 3\int_A^R \gamma_1(\tilde{R})\gamma_2^2(\tilde{R})\tilde{R}^2\, d\tilde{R}\right)^{1/3}}{\gamma_2 R}. \tag{2}$$

Hyperelasticity implies that there exists a strain energy function $W = W(\mathbf{A})$. Letting $\mathbf{T}$ denote the Cauchy stress tensor, the stress-strain relation is $\mathbf{T} = \mathbf{A}W_{\mathbf{A}} - p\mathbf{1}$, where p is a Lagrange multiplier representing hydrostatic pressure and $W_{\mathbf{A}}$ is the tensorial derivative of W w.r.t. $\mathbf{A}$. Denote the non-vanishing components of the Cauchy stress tensor by $t_1 = T_{11}$, the radial stress, and $t_2 = T_{22} = T_{33}$, the hoop stress. In terms of these variables, the stress-strain relationship is $t_1 = \alpha_1 W_1 - p$, $t_2 = \alpha_2 W_2 - p$, where $W_i = \frac{\partial W}{\partial \alpha_i}$. Mechanical equilibrium requires $\mathrm{div}(\mathbf{T}) = 0$, where div is the divergence in the current configuration - the only non-vanishing equation is

$$\frac{\partial t_1}{\partial r} + \frac{2}{r}(t_1 - t_2) = 0, \tag{3}$$

and from this we obtain a closed equation for the radial stress (see [3] for details):

$$\frac{\partial t_1}{\partial r} = \frac{\alpha}{r}\hat{W}'(\alpha). \tag{4}$$

Here we have introduced the auxiliary function $\hat{W}(\alpha) = W(\alpha^{-2}, \alpha, \alpha)$. Integrating Equation (4) and defining $P = t_1(A) - t_1(B)$ as the applied load on the shell, we have

$$-P = \int_a^b \frac{\alpha \hat{W}'}{r}\, dr. \tag{5}$$

This may be rewritten in terms of R as

$$-P = \int_A^B \frac{\hat{W}'(\alpha)\gamma_1}{\alpha^2 \gamma_2 R}\, dR. \tag{6}$$

2.2 Formulation of the problem

Given growth functions $\gamma_i(R)$ and an applied load P, Equation (6) may be thought of as defining a relation to solve for the inner radius a, since the outer radius b is a function of a via Equation (1) with $R = B, r = b$. Note that a is implicitly embedded in (6) through Equation (2). Once a is known, the deformation is completely determined. The question of void collapse is then: *what are the conditions on the growth functions $\gamma_i(R)$ and external pressure so that there exists a solution of mechanical equilibrium with $a = 0$?*

$a = 0 \Rightarrow$ complete collapse

Void collapse depends on the convergence of the integral (6) in the limit $a \to 0$. If the integral converges, then the value of the integral gives the applied load necessary for anticavitation. Mathematically, the difficulty is apparent by setting $a = 0$ in the representation (5). Physically, the issue is that sending $a \to 0$ requires zero circumferential stretch ($\alpha \to 0$ in (2)), or equivalently infinite radial stretch, at $R = A$.

The response of the shell to a given growth law, and thus whether or not anticavitation can be achieved, depends on the material properties, encompassed in the form of the strain energy function $\hat{W}$. There are many different strain energy functions available in the literature. In cavitation problems, Ball showed that it is the behavior of $\hat{W}(\alpha)$ as $\alpha \to \infty$ that determines whether the material admits cavitation. In contrast, Abeyaratne and Hou showed that the behavior of $\hat{W}$ as $\alpha \to 0$ determines void collapse. Here we show that with the addition of growth, the behavior of the auxiliary function both at zero and infinity plays a role in anticavitation. We classify the possibility of anticavitation for strain energy functions $\hat{W}$ with power law behavior as α approaches zero and infinity. Here, we use the notation

$$f(x) \sim g(x) \;\text{ as } x \to x_0 \text{ if } \lim_{x \to x_0} \frac{f(x)}{g(x)} = c \neq 0, \;\; c \text{ a constant}, \tag{7}$$

and restrict our attention to strain energy functions which satisfy the following hypothesis:

Hypothesis 1 *The auxiliary function $\hat{W}(\alpha)$ associated with the strain energy function W for an incompressible, hyperelastic material satisfies*

$$\hat{W}(\alpha) \sim \alpha^{m_0} \text{ as } \alpha \to 0, \;\; \hat{W}(\alpha) \sim \alpha^{m_\infty} \text{ as } \alpha \to \infty \tag{8}$$

for some constants m_0 and m_∞.

2.3 Finite growth

Along with the behavior of the strain energy, we will need to characterize the behavior of the growth functions γ_i. Define a *finite growth function* as being finite, continuous, and strictly positive. As a first case, we have the following:

Proposition 1 *A material undergoing finite growth admits spherically symmetric anticavitation if and only if $\hat{W}(\alpha)$ is bounded as $\alpha \to 0$.*

Proof: *Note that from (2), as r goes to zero so does α. By assumption, $\hat{W} \sim \alpha^{m_0}$ as $\alpha \to 0$. Then*

$$\frac{\alpha \hat{W}'}{r} \sim \frac{\alpha \alpha^{m_0-1}}{r} \sim \frac{1}{r^{1-m_0}} \text{ as } r \to 0. \tag{9}$$

Hence,

$$\int_0^b \frac{\alpha \hat{W}'}{r}\, dr \tag{10}$$

will converge if and only if $m_0 > 0$, i.e. if and only if $\hat{W}$ is bounded as $\alpha \to 0$.

The condition that $\hat{W}$ must be bounded as $\alpha \to 0$ coincides with the condition for void collapse given by Abeyaratne and Hou in the absence of growth. Hence, imposing finite growth cannot change the property of a material to admit anticavitation. However, for materials that satisfy this condition, void collapse can occur in the absence of external loading but through growth-induced residual stress. To illustrate this phenomenon, consider a material with strain energy $W = \frac{\mu}{n\beta}[\left(1+\beta(\alpha_1^2+\alpha_2^2+\alpha_3^2-3)\right)^n - 1]$, where $\mu > 0$, $\beta > 0$, and n are material constants as suggested in [1]. This strain energy function admits anticavitation for $n < 0$. We consider homogeneous, anisotropic growth for an unloaded shell by setting $\gamma_2 = 1$, $\gamma_1 > 1$ a constant, and the external pressure $P = 0$. Figure 1 plots the inner radius a as a function of γ_1 for initial radii $A = 1$, $B = 2$, and parameters $\mu = \beta = 1$. There is a critical value of γ_1 for which the cavity closes fully. Above this value, the inner radius is still zero, with a finite compressive radial stress induced at the origin due to the contact. Figure 1 shows the residual radial stress at the points marked I and II. The stress is greater in the cavity-filling solution II, and is non-zero at the center of the solid sphere ($R = 1$).

3 Diverging/vanishing γ_i

The condition that $\hat{W}$ be bounded as α goes to zero is very restrictive. For most models of materials in the literature (including neo-Hookean, Mooney-Rivlin, Ogden, and Varga), with $\hat{W}$ unbounded as $\alpha \to 0$, the integral in Equation (6) (or equivalently (5)) is divergent implying that anticavitation requires infinite external pressure. However, we have assumed that the functions γ_i are finite and strictly positive. As is discussed in Appendix A, divergent or vanishing growth functions can arise naturally in a cumulative growth process where growth is localised close to the boundary. Therefore, we consider materials for which the strain energy is unbounded as α approaches zero, and explore whether anticavitation is possible with diverging and/or vanishing γ_i.

Following the assumptions and notations of the previous section, we define the function

$$G(a) := \int_A^B \frac{\partial t_1}{\partial R}(R;a)\, dR = \int_A^B \frac{\hat{W}'(\alpha)\gamma_1}{\alpha^2 \gamma_2 R}\, dR. \tag{11}$$

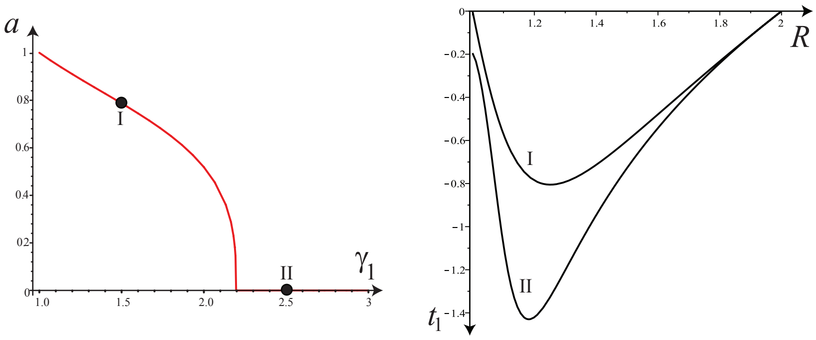

Fig. 1 Anticavitation of a shell (with initial radii $A = 1$ and $B = 2$) induced by radial growth. On the left, the inner radius in the current configuration, a, is plotted as a function of the radial growth γ_1 for $\gamma_2 = 1$, $\mu = \beta = 1$, $n = -1$. The radial stress corresponding to the points marked I, II is shown on the right. The radial growth creates a compressive radial stress.

For a given external pressure P, the positive root of $G(a) + P = 0$ gives the inner radius. To focus on the effect of growth alone, we consider shells with no external pressure. Hence, we set $P = 0$ and explore the convergence or divergence of the integral $G(0)$. We allow the functions γ_i to either blow up or vanish at the inner boundary, and consider whether this characteristic of the growth functions can change the divergent nature of the integral $G(0)$. In particular, we define $y := R - A$ and assume that as $y \to 0$ (that is $R \to A$) we have

$$\gamma_1 \sim y^{-p_1}, \quad \gamma_2 \sim y^{-p_2}, \tag{12}$$

where p_1 and p_2 are real numbers. From (1) with $a = 0$, we find that

$$r^3 \sim y^{1-p_1-2p_2} \text{ as } R \to A. \tag{13}$$

For the map $r(R)$ to be well defined, we therefore require that $1 - p_1 - 2p_2 > 0$. Next, using the relation $\alpha = r/(\gamma_2 R)$, we have $\alpha \sim y^{(1-p_1+p_2)/3}$. The behavior of $G(0)$ will depend on whether α is vanishing or diverging as $R \to A$. Thus, from here we divide the problem into 3 cases, based on the sign of the quantity

$$J := 1 - p_1 + p_2. \tag{14}$$

3.0.1 Case 1: $J > 0$

If $J > 0$, then $\alpha \to 0$ as $R \to A$. Since by assumption $\hat{W} \sim \alpha^{m_0}$ as $\alpha \to 0$, $m_0 < 0$. In this case, we have

$$\frac{\partial t_1}{\partial R} \sim y^{E_1(m_0)} \text{ as } y \to 0. \tag{15}$$

where $E_1(m) = \frac{J}{3}(m-3) - p_1 + p_2$. The requirement for the boundedness of the radial stress is $E_1(m_0) > -1$. This reduces to $Jm_0 > 0$ which cannot hold since $J > 0$ and $m_0 < 0$.

3.0.2 Case 2: $J<0$

For $J<0$, $\alpha \to \infty$ as $R \to A$, and we consider the behavior of $\hat{W}$ as $\alpha \to \infty$. For isotropic material, W has the symmetry $W(\alpha_1,\alpha_2,\alpha_2)=W(\alpha_2,\alpha_1,\alpha_1)$. Thus for $\hat{W}(\alpha)=W(\alpha^{-2},\alpha,\alpha)$, it follows that $\lim_{\alpha\to 0}\hat{W}(\alpha)=\lim_{\alpha\to\infty}\hat{W}(\alpha)=\infty$, and we conclude that $\hat{W}$ is unbounded at infinity. Therefore, if $\hat{W}\sim\alpha^{m_\infty}$ as $\alpha\to\infty$, it follows that $m_\infty>0$. Similar to Case 1,

$$\frac{\partial t_1}{\partial R} \sim y^{E_1(m_\infty)} \text{ as } y \to 0, \tag{16}$$

and convergence requires $E_1(m_\infty)>-1$, which simplifies to $Jm_\infty>0$ which again does not hold based on the assumptions of J and m_∞.

3.0.3 Case 3: $J=0$

If $J=0$, α and $\hat{W}'(\alpha)$ are constants as $R\to A$ and hence play no role in the convergence or divergence of the integral. Here,

$$\frac{\partial t_1}{\partial R} \sim y^{-p_1+p_2} \text{ as } y \to 0, \tag{17}$$

but the requirement for convergence is equivalent to requiring $J>0$, which does not hold.

From these 3 cases, we conclude that even if we allow the growth functions to diverge or vanish at the inner boundary, we still cannot send the inner radius a to zero. This is quite in contrast to elastic cavitation, where a void may be formed with finite growth and no external pressure [21], and highlights the notion that cavitation and anticavitation are not "inverse processes" in a mathematical sense. To summarize:

Proposition 2 *Let $\gamma_i \sim (R-A)^{-p_i}$ as $R\to A$, $i=1,2$, be radial growth functions. If $\hat{W}$ is unbounded as $\alpha\to 0$, the material does not admit anticavitation for any real numbers p_1 and p_2.*

3.1 The p_1-p_2 plane: filling the void

The previous proposition seems to point to the impossibility of anticavitation in locally unbounded strain energy-functions. However, in this section we show that the void can be made arbitrarily small with the proper balance of diverging/vanishing growth. To do this, we consider more closely the relationship between the rates of growth, captured by the exponents p_1 and p_2 in the relation (12). Figure 2 depicts the p_1-p_2 plane. The requirement for a well defined map $r(R)$ is that $p_1<1-2p_2$, which divides the plane into a valid and invalid region. We further divide the plane based on the sign of the quantity J. Underlying Proposition 2 is the fact that for any choice of p_1 and p_2, $G(0)$ is a diverging integral. The following result establishes the direction of divergence of this integral (plus or minus infinity) .

Proposition 3 *Assume that $\hat{W}(\alpha)$ is unbounded as $\alpha\to 0$. Then, for $J\neq 0$,*

$$\lim_{a\to 0^+} G(a) = -\operatorname{sgn}(J)\infty. \tag{18}$$

That is, the direction of divergence (plus or minus infinity) of the integral $G(a)$ given in Equation (11) in the limit $a=0$ is determined by the sign of J.

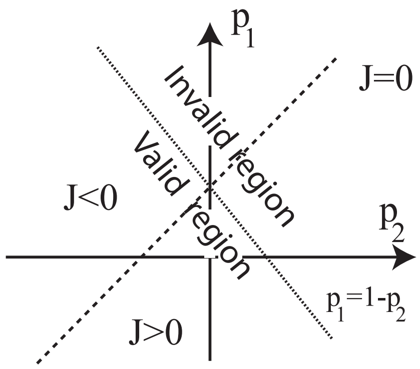

Fig. 2 Depiction of p_1-p_2 plane. The line $p_1 = 1-2p_2$ divides the plane into a valid and invalid region. The plane is further divided based on the sign of the quantity J.

Proof: *The proof comes from an analysis of the sign of $\hat{W}'(\alpha)$ as $a \to 0$. We use again the symmetry of W for isotropic materials: $W(\alpha_1, \alpha_2, \alpha_2) = W(\alpha_2, \alpha_1, \alpha_1)$, as well as the fact that the strain energy W is non-negative for all deformations. Thus for $\hat{W}(\alpha) = W(\alpha^{-2}, \alpha, \alpha)$, we have $\lim_{\alpha\to 0} \hat{W}(\alpha) = \lim_{\alpha\to\infty} \hat{W}(\alpha) = \infty$. It follows that*

$$\lim_{\alpha\to 0} \hat{W}'(\alpha) = -\infty, \quad \lim_{\alpha\to\infty} \hat{W}'(\alpha) = \infty.$$

Since $\alpha \to 0$ as $R \to A$ when $J > 0$ and $\alpha \to \infty$ as $R \to A$ when $J < 0$ and since all other terms in $G(0)$ are non-negative, the integral $G(0)$ will be negative when $J > 0$ (i.e. $\alpha \to 0$) and positive when $J < 0$ ($\alpha \to \infty$). The result immediately follows.

Given this change in divergence based on the sign of J, the following theorem establishes that a solution may be found with an arbitrarily small void.

Theorem 1 *Let $\hat{W}(\alpha)$ be unbounded as $\alpha \to 0$, and assume that the growth functions are such that $\gamma_i \sim (R-A)^{-p_i}$ as $R \to A$. Then for any fixed $p_2 < 0$ and $\varepsilon > 0$, there exists p_1 for which the unloaded spherical shell has an inner radius $a < \varepsilon$.*

Proof: *Let $\varepsilon > 0$ and $p_2 < 0$ be fixed. Since the function $G(a)$ depends on the exponent p_1, we define for each p_1 the function $H(a, p_1) = G(a)$. Further, there is a distinguished exponent $p_1^* = 1 + p_2$ for which $J = 0$. Then, following Case 3 above, we have,*

$$\lim_{a\to 0^+} H(a, p_1^*) = s(p_2)\infty \tag{19}$$

where $s(p_2)$ is either $+1$ or -1 depending on the value of p_2 and the strain-energy function. Without loss of generality, we consider here the case where $s(p_2) = +1$, as the case where $s(p_2) = -1$ follows with appropriate sign changes. Since H is continuous with respect to a, we can choose $0 < a_1 < \varepsilon$ such that $\phi := H(a_1, p_1^*) > 0$. For fixed $a_1 > 0$, H is a continuous function of p_1. Hence there exists $\delta > 0$ such that

$$|p_1^* - p_1| < \delta \Rightarrow |H(a_1, p_1^*) - H(a_1, p_1)| < \phi. \tag{20}$$

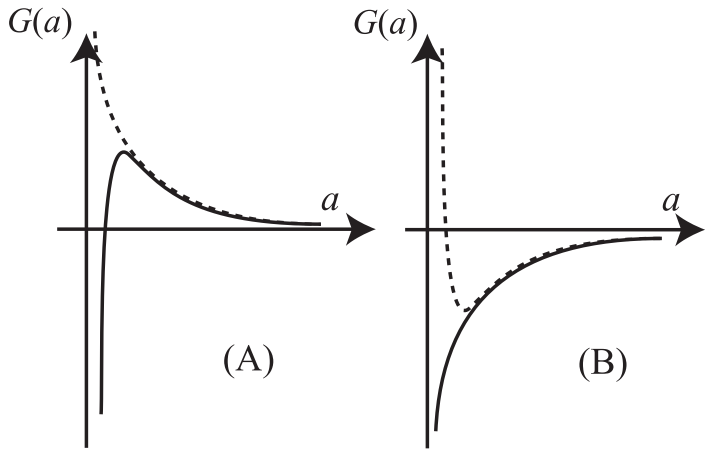

Fig. 3 Depiction of the local behavior of $G(a)$ near $a = 0$. The dashed (solid) lines represent $G(a)$ for (p_1, p_2) directly above (below) $J = 0$. The change in sign of the divergence ensures a root arbitrarily close to zero. Referring to Theorem 1, in (A) $s(p_2) = +1$ and $s(p_2) = -1$ in (B). Thus in (A), an arbitrarily small root appears as (p_1, p_2) approach $J = 0$ from below. In (B), an arbitrarily small root appears as (p_1, p_2) approach $J = 0$ from above.

Let $\hat{p}_1 = p_1^* - \delta/2$; then $H(a_1, \hat{p}_1) > 0$. Further, since $\hat{p}_1 < p_1^*$, $J > 0$ at the point $(\hat{p}_1, p2)$ (see Figure 2), and so

$$\lim_{a \to 0^+} H(a, \hat{p}_1) = -\infty. \tag{21}$$

Again by continuity of H as a function of a, there exists $a^* > 0$ such that $H(a, \hat{p}_1) < 0$ $\forall a < a^*$. Therefore, we can choose $0 < a_2 < a_1$ such that $H(a_2, \hat{p}_1) < 0$ and by the Intermediate Value Theorem there exists $a_3 \in (a_2, a_1)$ such that $H(a_3, \hat{p}_1) = 0$. Since by construction $a_3 < \varepsilon$, the result follows.

The above argument is illustrated in Figure 3, where the local shape of the curve $G(a)$ near $a = 0$ will transition according to one of two possibilities. In Figure 3, the dashed lines represent the shape of the curve for (p_1, p_2) located directly above the line $J = 0$ (i.e. in the $J < 0$ region of Figure 2) and diverge to infinity, while the solid lines represent the curve for (p_1, p_2) directly below $J = 0$, with the divergence to minus infinity. In Case (A), the change of divergence as (p_1, p_2) cross the line $J = 0$ results in a root appearing arbitrarily close to $a = 0$ when (p_1, p_2) are arbitarily close to but below $J = 0$. In other words, in this case for fixed $p_2 < 0$, taking p_1 less than but arbitrarily close to $p_1^* = 1 + p_2$ results in a solution of mechanical equilibrium existing with arbitrarily small void. In (B), an arbitrarily small root appears for (p_1, p_2) arbitrarily close to but above $J = 0$, that is for p_1 greater than but arbitrarily close to p_1^*. In either case, the general result is that an arbitrarily small root exists for some (p_1, p_2) close to the line $J = 0$.

Which scenario is appropriate, i.e. whether (p_1, p_2) should approach $J = 0$ from above or below to obtain the arbitrarily small root, depends on the behavior of $G(0)$ *on* the line $J = 0$. When $J = 0$, α approaches a constant as $R \to A$, and the direction of the divergence will depend on the particular strain energy function and can also change along the line $J = 0$. If, at a particular point on the line $J = 0$, $G(0)$ diverges to positive infinity, then that point corresponds to Case (A), and the root may be sent to zero by approaching $J = 0$ from below (i.e. the limit of $p_1 \to p_1^{*-}$ for fixed p_2). To see this, consider: in Case (A), the divergence on $J = 0$ must be to positive infinity; otherwise, if $G(0)$ diverged to negative infinity either the root was not able to be made arbitrarily small, contradicting the continuity, or else the root became identically zero, contradicting the divergence. If instead $G(0)$ diverges to negative infinity, the point corresponds to Case (B) and $J = 0$ should be approached from above (the limit $p_1 \to p_1^{*+}$ for fixed p_2).

3.2 Physical interpretation

In view of the valid/invalid division of the plane in Figure 2, a consequence of the above results is that p_2 must be negative to send a root to zero, which physically corresponds to infinite circumferential resorption at the cavity surface. Recall, however, that the actual *form* of growth depends on the ratio γ_1/γ_2. Since we do not prescribe the exact form of the γ_i, only their asymptotic behavior, we cannot compute this ratio. We can, however, infer that the region $J > 0$ corresponds to infinite circumferential growth at the cavity surface, since in this region the elastic response to the growth is an infinite radial elastic stretch ($\alpha \to 0$ as $R \to A$), whereas in the region $J < 0$ there is infinite circumferential elastic stretch in response to the growth, and thus the form of growth must be radial at the cavity surface. We see, then, that an arbitrarily small void is achieved by approaching the boundary between the two regions of infinite radial and circumferential growth.

The question still arises, though, of why the conditions that lead to anticavitation are so restrictive in comparison with cavitation. First, note that the difficulty is essentially mathematical rather than physical. In a physical sense, there is almost no difference between a sphere with an arbitrarily small void and a solid sphere, and so in some instances it would be fair to say that anticavitation has occurred physically, even though a singularity cannot be overcome mathematically. This distinction still does not really answer the question. In the context of growth, the fundamental difference between cavitation and anticavitation seems to be in the role of residual stress. When a solid sphere undergoes differential growth, growth causes a build-up of stress at the origin, which eventually causes the formation of a void [21]. A shell undergoing differential growth also builds up residual stress, but in this case the residual stress at the cavity surface *resists* the closing of the void, since the stress increases with descreasing cavity radius. Thus, residual stress enhances cavitation, but hinders anticavitation.

4 Stability

We have shown that a root of $G(a)$ may be brought arbitrarily close to zero. This means that there exists a mechanical solution of the problem with an arbitrarily small void. However, we have not demonstrated that these solutions can be reached and are stable. In the transition crossing the line $J = 0$, multiple roots may occur, only one of which may be stable. A stability analysis for symmetric deformations can be performed by considering the minima of the potential energy.

Theorem 2 *Let $a > 0$ be a solution of $G(a) = 0$ corresponding to the inner radius of an equilibrium solution in the set of radially symmetric deformations. Then, the associated deformation is locally stable if $G'(a) > 0$, and locally unstable if $G'(a) < 0$.*

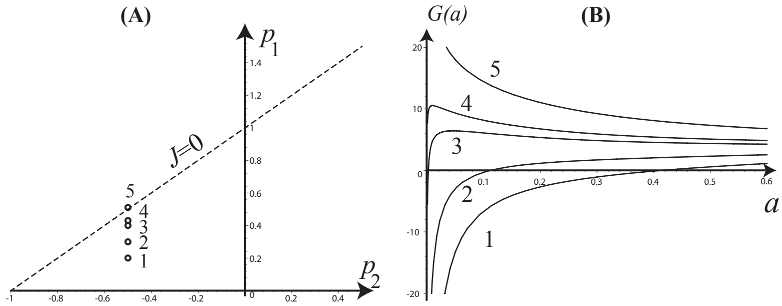

Fig. 4 (A) 5 points in the p_1-p_2 plane defining 5 choices of γ_i. (B) The corresponding curves $G(a)$ for a neo-Hookean material. Anticavitation is achieved as $J = 0$ is approached from below.

Proof: *The potential energy for the deformation, as a function of the inner radius a, is found by integrating the strain energy function over the shell; that is*

$$U(a) = \int_a^b \hat{W}(\alpha) 4\pi r^2 \, dr = \int_A^B \hat{W}(\alpha)\gamma_1\gamma_2^2 4\pi R^2 \, dR. \tag{22}$$

Taking a derivative with respect to a, and using (1), (2), and the definition of $G(a)$ (11), we obtain

$$U'(a) = 4\pi a^2 G(a). \tag{23}$$

Taking a second derivative, and using the fact that $G(a) = 0$ at a point of equilibrium, we have $U''(a) = 4\pi a^2 G'(a)$, and the result follows by the concavity of the energy potential.

Whether or not an arbitrarily small void can be physically achieved depends on whether or not there are multiple roots, which depends on the form of the curve $G(a)$, which in turn depends on the choice of strain energy function. As an example, consider a neo-Hookean material, characterized by a strain energy function $W = \mu(\alpha_1^2 + \alpha_2^2 + \alpha_3^2 - 3)$, where $\mu > 0$, so that $\hat{W} = \mu(\alpha^{-4} + 2\alpha^2 - 3)$.

In Figure 4, the curve $G(a)$ is plotted for a sequence of points in the p_1-p_2 plane, where $\gamma_1 = (R-A)^{-p_1}$, $\gamma_2 = (R-A)^{-p_2}$, and parameters $A = 1$, $B = 2$, and $\mu = 1$. The 5 curves in (B) correspond to the 5 points in (A). As the line $J = 0$ is approached from below, the single root of the curve approaches zero. Since there is only a single root, anticavitation is possible in the limit of p_1 approaching 0.5 from the left (in this example $p_2 = -0.5$ is fixed). Above the line (point 5), the divergence at zero flips, and the root disappears. The radial residual stress at points 3 and 4, when the cavity radius is nearly zero, is compressive right at the cavity surface, but has a very sharp transition so that it is tensile over most of the shell (not plotted). As mentioned before, this build-up and sharp gradient in stress serves to resist the full collapse of the void.

In Figure 5, $p_2 = -1$ is fixed. In (A), $G(a)$ is plotted on both sides of $J = 0$. For $p_1 = -0.06$, there is a root, so that crossing the line $J = 0$ to the point $p_1 = 0.06$ yields a second root, an unstable solution based on Theorem 2 and the slope of the curve $G(a)$ at point a_2. In (B), the residual radial stress is plotted for each of these two roots, a_1 and a_2. For both solutions, the residual stress is tensile, and as expected, the stress is much higher in the unstable solution with root a_2.

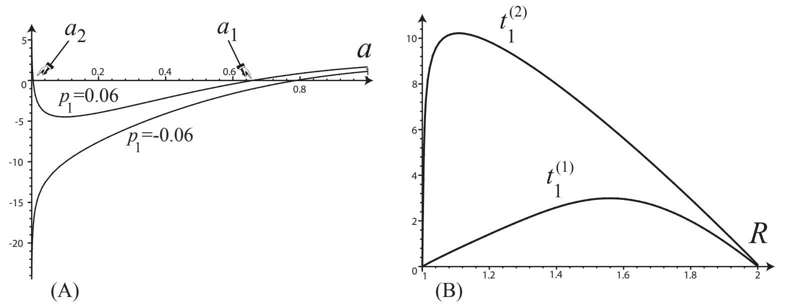

Fig. 5 (A) The curve $G(a)$ above ($p_1 = 0.06$) and below ($p_1 = -0.06$) $J = 0$ for $p_2 = -1$. (B) The radial stress for the two roots a_1 and a_2 from (A).

Comparing Figures 4 and 5, a couple things are worth noting. When $p_2 = -0.5$, $G(0)$ diverges to positive infinity and has no root, whereas with $p_2 = -1$, $G(0)$ diverges to negative infinity and has a single root. This means that following a path along $J = 0$ from $p_2 = -0.5$ to $p_2 = -1$, there are two interesting bifurcations. At the point $p_2 \approx -0.795$, $G(a)$ tangentially touches the axis but still diverges to $+\infty$, so that beyond this point there are two roots of $G(a)$ on $J = 0$. A second bifurcation occurs right at $p_2 = -1$, at which point the divergence of $G(0)$ switches to $-\infty$. Thus, for $p_2 \in (-1, -0.795)$, there are two roots on $J = 0$, so that a third arbitrarily small root is added by approaching $J = 0$ from below. Based on Theorem 2, the small root and the largest root are both stable, although the radial residual stress is higher for the void-filling smaller root. By a continuous change of parameter, the larger root is the continuation of the path and will be the one that is observed. Hence, for for all $p_2 \in (-1, -0.795)$, anticavitation (in the arbitrarily small void sense) is physically unattainable.

It is worth noting that the neo-Hookean material used in this example does not satisfy the criterion of Abeyaratne and Hou for anticavitation given in [1], i.e. $\hat{W}$ is unbounded as $\alpha \to 0$. In other words, external pressure on the shell cannot cause void collapse, whereas growth can cause the void to become arbitrarily small.

5 Conclusion

In this paper, we explored anticavitation in a growing elastic shell as a function of the form and of the rate of imposed growth. The question of whether void collapse is possible is formulated in terms of the convergence or divergence of a single integral, given in Equation (5). Our analysis consists in determining radial growth functions for which convergence of this integral is achieved. We concluded that the integral will always diverge, even for growth functions which diverge or vanish at the point where the integral is improper. Nevertheless, based on the rates at which the growth functions diverge/vanish, the void may still be made arbitrarily small. This can only be achieved in the proper limits of the rates at which the growth functions diverge/vanish, illustrating the fine balance of radial and circumferential growth needed for anticavitation. Note, however, that we have only considered strain energy functions with power law behavior. An extension of the present work would be to study functions $\hat{W}(\alpha)$ which are either bounded by power laws or diverge faster than any power, in which case it is likely that exponentially diverging/vanishing growth functions would also be necessary.

The effect of growth on material properties is clear – arbitrarily small voids can be achieved through growth alone in materials for which an arbitrarily small void requires a limitless increase in pressure when growth is not included. Furthermore, the analysis presented in the Appendix suggests that diverging or vanishing growth functions naturally arise in the context of cumulative growth. Physically, such functions appear as the result of (finite) material growth being focused on the cavity surface and accumulating; multiple steps of "incremental growth" then become mathematically similar to the diverging/vanishing functions we have considered.

The present study of void collapse reveals that completely filling a void through a purely elastic mechanism in physical or biological systems is mathematically impossible, yet a delicate balance between growth laws and material parameters *can* lead to arbitrarily small voids. In practice, complete anticavitation would involve either a control mechanism or other mechanisms such as accretion at the inner boundary or asymmetric deformation of the body. In particular, as compressive forces increase, a buckling instability could be triggered, as is shown in cylindrical tubes in [55]. Nonetheless, from a mechanical standpoint, the simplified model presented here illustrates the important role of differential growth in anticavitation processes.

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Appendix: Cumulative growth

We have analyzed the growth of a spherical shell with radial growth functions which either diverged or vanished at the inner edge $R = A$. As a single growth step, this may appear unphysical. However, this single step should be seen as the result of a cumulative growth process. In a continuously growing body, growth can be modeled by a relation of the form

$$\dot{\mathbf{G}} = H(\mathbf{G}, \mathbf{A}, \mathbf{T}, \ldots, ; \mathbf{X}, t); \tag{24}$$

where the rate of growth is not necessarily constant, and might in general depend on the growth and strain tensors, stress, position, time, and potentially other factors. This can be approximated by a discrete growth process, where (24) is replaced by $\mathbf{G}(t + \Delta t) = \mathbf{G}(t) + \Delta t H$. This relation may be seen to define an incremental growth, $\mathbf{G}_{\text{inc}} := \mathbf{G}(t + \Delta t) - \mathbf{G}(t)$. If, at each incremental step, the elastic response is captured by $\mathbf{A}_{\text{inc}}$, then through multiplicative decomposition we can define an incremental deformation $\mathbf{F}_{\text{inc}} = \mathbf{A}_{\text{inc}} \cdot \mathbf{G}_{\text{inc}}$. Thus, at each step, the material grows according to some incremental growth law, followed by an elastic response necessary for compatibility. For each incremental step, one can define a total growth tensor, such that the i^{th} incremental step is equivalent to a single step with a deformation tensor defined from the initial, stress free configuration (see [19] for details).

In the spherical geometry we have considered here, let R be the radius in the initial configuration and r_{i-1} the radius in the current configuration after the $(i-1)^{\text{th}}$ step. Then the incremental growth tensor for the i^{th} step will be of the form

$$\mathbf{G}_{\text{inc}}^{(i)} = \text{diag}\left(\gamma_1^{(i)}(r_{i-1}),\ \gamma_2^{(i)}(r_{i-1}),\ \gamma_2^{(i)}(r_{i-1})\right). \tag{25}$$

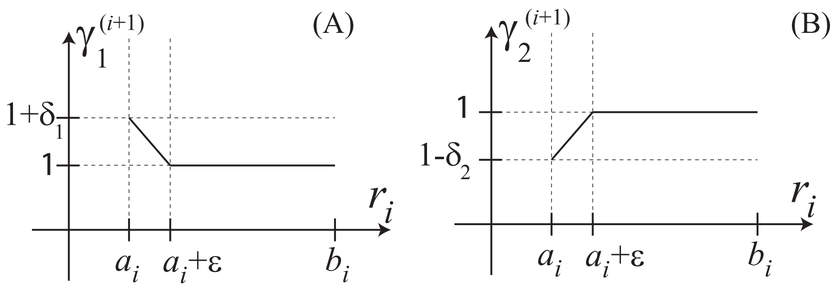

Fig. 6 Piecewise incremental growth laws in a cumulative growth process.

The total deformation defines a map $r_{i-1} = r_{i-1}(R)$, and so the γ are functions of the radius in the initial configuration, R. The *total* growth tensor is [19]

$$\mathbf{G}^{(i)} = \operatorname{diag}\left(\Pi_{n=1}^{i}\gamma_1^{(n)}(R),\ \Pi_{n=1}^{i}\gamma_2^{(n)}(R),\ \Pi_{n=1}^{i}\gamma_2^{(n)}(R)\right). \tag{26}$$

If the shell is growing in a continuous process such that growth is always focused on the inner surface, then the incremental γ will take extreme values at the inner edge, corresponding to $R = A$. Therefore, the total growth functions, which are products of the growth functions of all previous steps, will magnify this effect and will approach either zero or infinity at the inner surface. Thus, the total growth functions would be well approximated by the diverging/vanishing γ_i we have considered in this paper.

In this process, the possibility of anticavitation will depend on the specific form of the incremental growth laws, encompassed by a relation of the form (24). As a simple example, consider a neo-Hookean material subject to the piecewise linear growth functions pictured in Figure 6.

At each incremental step, the region over which growth/resorption is restricted is given by the parameter ε, while the δ_i dictate the rate or amount of growth/resorption. Figure 7 shows the results of the cumulative growth process for the parameters $\delta_1 = 0.6$, $\delta_2 = 0.3$, and $\varepsilon = 0.5$. Here, the balance of radial growth and circumferential resorption is such that the growth collapses the void, as seen in Figure 7(A). The total growth function $\gamma_1(R)$ at the 9^{th} step is plotted in Figure 7(B), illustrating the blow-up like behavior of the radial growth function after several steps.

As would be expected from the analysis of Section 3, a small change in the growth rate parameters δ_i can cause anticavitation to be no longer possible (plots not included).

References

1. R. Abeyaratne and H-S Hou. Void collapse in an elastic solid. *J. Elasticity*, 26:23–42, 1991.
2. P. W. Alford, J. D. Humphrey, and L. A. Taber. Growth and remodeling in a thick-walled artery model: effects of spatial variations in wall constituents. *Biomech. Model Mechanobiol.*, 7:245–262, 2007.
3. M. Ben Amar and A. Goriely. Growth and instability in elastic tissues. *J. Mech. Phys. Solids*, 53:2284–2319, 2005.
4. D. Ambrosi and L. Preziosi. Cell adhesion mechanisms and stress relaxation in the mechanics of tumours. *Biomechanics and Modeling in Mechanobiology*, 8(5):397–413, 2009.
5. J. M. Ball. Discontinuous equilibrium solutions and cavitation in nonlinear elasticity. *Phil. Trans. Roy. Soc. London A*, 306:557–610, 1982.

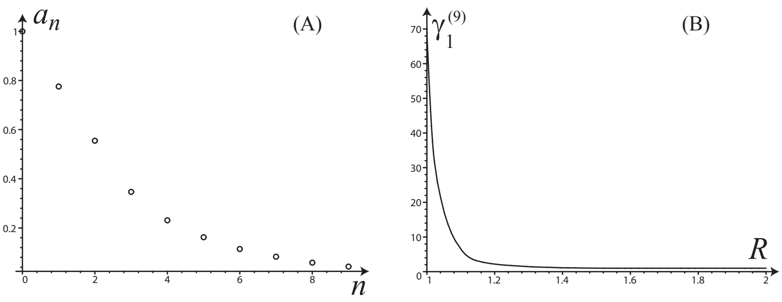

Fig. 7 Cumulative growth process in a neo-Hookean shell. (A) - Progression of the inner radius through 9 steps of growth. (B) The total radial growth function for the 9$^{\text{th}}$ step of growth, the function is developing singular type behavior at the inner surface $R = 1$. The incremental growth functions are as pictured in Figure 6 with $\delta_1 = 0.6$, $\delta_2 = 0.3$, and $\varepsilon = 0.5$.

6. G. K. Batchelor. *An Introduction to Fluid Mechanics*. Cambridge University Press, Cambridge, 1967.
7. H. G. Burstrom. Tissue Tensions during Cell Elongation in Wheat Roots and a Comparison with Contractile Roots. *Physiologia Plantarum*, 25(3):509–513, 1971.
8. L. Cardamone, A. Valentin, J.F. Eberth, and J.D. Humphrey. Origin of axial prestretch and residual stress in arteries. *Biomechanics and Modeling in Mechanobiology*, 8(6):431–446, 2009.
9. J. Casey and P. M. Naghdi. A remark on the use of the decomposition $F = F_e F_p$ in plasticity. *Trans. ASME*, 47:672–675, 1980.
10. H. R. Chaudhry, B. Bukiet, A. Davis, A. B. Ritter, and T. Findley. Residual stresses in oscillating thoracic arteries reduce circumferential stresses and stress gradients. *Journal of biomechanics*, 30(1):57–62, 1997.
11. Y. Chen and A. Hoger. Constitutive functions of elastic materials in finite growth and deformation. *J. Elasticity*, 59, 2000.
12. A. Delfino, N. Stergiopulos, J. E. Moore, and J. J. Meister. Residual strain effects on the stress field in a thick wall finite element model of the human carotid bifurcation. *J. Biomech.*, 30(8):777–786, 1997.
13. JM Dinwoodie. Growth stresses in timber–a review of literature. *Forestry*, 39(2):162, 1966.
14. J. Dollhofer, A. Chiche, V. Muralidharan, C. Creton, and C. Y. Hui. Surface energy effects for cavity growth and nucleation in an incompressible neo-hookean material - modeling and experiment. *Int. J. Solids Structures*, 41:6111–6127, 2004.
15. C. Fond. Cavitation criterion for rubber materials: a review of void-growth models. *J. Polymer Sci. B: Polymer Phys.*, 39:2081–2096, 2001.
16. Y. C. Fung. What are the residual stresses doing in our blood vessels? *Annals of Biomedical engineering*, 19(3):237–249, 1991.
17. K. Garikipati. The Kinematics of Biological Growth. *Applied Mechanics Reviews*, 62:030801, 2009.
18. A. N. Gent and P. B. Lindley. Internal rupture of bonded rubber cylinders in tension. *Proc. Roy. Soc. London A*, 249:195–205, 1958.
19. A. Goriely and M. Ben Amar. On the definition and modeling of incremental, cumulative, and continuous growth laws in morphoelasticity. *Biomech. Model. Mechanobiol.*, 6(5):289–296, 2007.
20. A. Goriely and M. Ben Amar. Differential growth and instability in elastic shells. *Phys. Rev. Lett.*, 94(19):198103, 2005.
21. A. Goriely, D. E. Moulton, and R. Vandiver. Elastic cavitation and differential growth in biological tissues. *Preprint*, 2010.
22. A. Goriely and R. Vandiver. On the mechanical stability of growing arteries. *IMA J. Appl. Math.*, Accepted, 2010.
23. A. Guillou and R. W. Ogden. Growth in soft biological tissue and residual stress development. *Mechanics of biological tissue. Springer, Heidelberg*, pages 47–62, 2006.
24. A. Gupta, D. J. Steigmann, and J. S. Stölken. On the evolution of plasticity and incompatibility. *Mathematics and Mechanics of Solids*, 12(6):583–610, 2007.
25. D. M. Haughton. On non-existence of cavitation in incompressible elastic membranes. *Quarterly Journal of Mechanics and Applied Mathematics*, 39(2):289–296, 1986.

26. G. Himpel, E. Kuhl, A. Menzel, and P. Steinmann. Computational Modelling of Isotropic Multiplicative Growth. *Computer Modeling in Engineering and Sciences*, 8(2):119–134, 2005.
27. A. Hoger. On the determination of residual stress in an elastic body. *J. Elasticity*, 16, 1986.
28. G.A. Holzapfel and R.W. Ogden. Modelling the layer-specific three-dimensional residual stresses in arteries, with an application to the human aorta. *Journal of the Royal Society Interface*, 7(787-799), 2010.
29. C. O. Horgan and D. A. Polignone. Cavitation in nonlinearly elastic solids: A review. *Appl. Mech. Rev.*, 48(8):471–485, 1995.
30. B. E. Johnson and A. Hoger. The dependence of the elasticity tensor on residual stress. *J. Elasticity*, 33, 1993.
31. S. M. Klisch, T. J. Van Dyke, and A. Hoger. A theory of volumetric growth for compressible elastic biological materials. *Math. Mech. Solids*, 6:551–575, 2001.
32. O. Lopez-Pamies. Onset of cavitation in compressible, isotropic, hyperelastic solids. *J. Elasticity*, 94:115–145, 2009.
33. V. A. Lubarda and A. Hoger. On the mechanics of solids with a growing mass. *Int. J. Solids Struct.*, 39:4627–4664, 2002.
34. J. McMahon, A. Goriely, and M. Tabor. Spontaneous cavitation in growing elastic membranes. *Math. Mech. Solids*, doi:10.1177/1081286508092010(8):471–485, 2008.
35. PM Naghdi. A critical review of the state of finite plasticity. *Zeitschrift für Angewandte Mathematik und Physik (ZAMP)*, 41(3):315–394, 1990.
36. R. W. Ogden. *Non-linear Elastic Deformation*. Dover, New York, 1984.
37. T.J. Pence and H. Tsai. On the cavitation of a swollen compressible sphere in finite elasticity. *International Journal of Non-Linear Mechanics*, 40(2-3):307–321, 2005.
38. T.J. Pence and H. Tsai. Swelling induced cavitation of elastic spheres. *Math. Mech. Solids*, 11(5):527, 2006.
39. K. A. Pericak-Spector, J. Sivaloganathan, and S. J. Spector. An explicit radial cavitation solution in nonlinear elasticity. *Math. Mech. Solids*, 7(1):87–93, 2002.
40. W. S. Peters and A. .D Tomos. The history of tissue tension. *Ann Bot (Lond)*, 77(6):657–65, 1996.
41. D. A. Polignone and C. O. Horgan. Cavitation for incompressible anisotropic nonlinearly elastic spheres. *J. Elasticity*, 33:27–65, 1993.
42. E. K. Rodriguez, A. Hoger, and A. McCulloch. Stress-dependent finite growth in soft elastic tissue. *J. Biomechanics*, 27:455–467, 1994.
43. E. K. Rodriguez, A. Hoger, and A. McCulloch. Stress-dependent finite growth in soft elastic tissues. *J. Biomech.*, 27:455–467, 1994.
44. J. Sivaloganathan. On cavitation and degenerate cavitation under internal hydrostatic pressure. *Proc. Roy. Soc. London A*, 455:3645–3664, 1999.
45. J. Sivaloganathan and S. J. Spector. On cavitation, configurational forces and implications for fracture in a nonlinearly elastic material. *J. Elasticity*, 67:25–49, 2002.
46. R. Skalak, S. Zargaryan, R. K. Jain, P. A. Netti, and A. Hoger. Compatibility and the genesis of residual stress by volumetric growth. *J. Math. Biol.*, 34:Zargaryan, 1996.
47. L. A. Taber. Biomechanics of growth, remodeling and morphogenesis. *Appl. Mech. Rev.*, 48:487–545, 1995.
48. L. A. Taber. Biomechanical growth laws for muscle tissues. *J. Theor. Biol.*, 193:201–213, 1998.
49. L. A. Taber and D. W. Eggers. Theoretical study of stress-modulated growth in the aorta. *J. Theor. Biol.*, 180:343–357, 1996.
50. L. A. Taber and J. D. Humphrey. Stress-modulated growth, residual stress, and vascular heterogeneity. *J. Biomech. Eng.*, 123:528–535, 2001.
51. R. N. Vaishnav and J. Vossoughi. Residual stress and strain in aortic segments. *J. Biomech.*, 20(3):235–239, 1987.
52. R. Vandiver and A. Goriely. Tissue tension and axial growth of cylindrical structures in plants and elastic tissues. *Europhys. Lett. (EPL)*, 84(58004), 2008.
53. H. Xiao, T. O. Bruhns, and A. Meyers. Elastoplasticity beyond small deformations. *Acta Mechanica*, 182:31–111, 2006.
54. F. Yuan. Stress is good and bad for tumors. *Nature Biotechnology*, 15(8):722–723, 1997.
55. D. E. Moulton and A. Goriely. Circumferential buckling instability of a growing cylindrical tube. *Preprint*, 2010.