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Riemann–Cartan Geometry of Nonlinear Dislocation Mechanics

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Abstract

We present a geometric theory of nonlinear solids with distributed dislocations. In this theory the material manifold—where the body is stress free—is a Weitzenböck manifold, that is, a manifold with a flat affine connection with torsion but vanishing non-metricity. Torsion of the material manifold is identified with the dislocation density tensor of nonlinear dislocation mechanics. Using Cartan's moving frames we construct the material manifold for several examples of bodies with distributed dislocations. We also present non-trivial examples of zero-stress dislocation distributions. More importantly, in this geometric framework we are able to calculate the residual stress fields, assuming that the nonlinear elastic body is incompressible. We derive the governing equations of nonlinear dislocation mechanics covariantly using balance of energy and its covariance.

Contents

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1. Introduction

In continuum mechanics, one idealizes a body as a collection of material points, with each assumed to be a mathematical point. Kinematics of the body is then represented by a time-dependent placement of the material points, that is, by a time-dependent deformation mapping. It is then assumed that there exists a stress-free configuration (a natural configuration) that can be chosen as a reference configuration. This natural configuration is heavily used in the nonlinear mechanics of solids. Defects are known to be the source of many interesting properties of materials; in metals, dislocations are particularly important. A body with a distribution of dislocations and no external forces will develop internal stresses, in general. Thus, the initial configuration cannot be a reference configuration in the classical sense. In other words, if one cuts the undeformed configuration into small pieces and lets them relax, the resulting relaxed small pieces cannot fit together; that is, the relaxed configuration is not compatible when embedded in Euclidean space. However, one may imagine that the small relaxed material points lie in a non-Riemannian manifold with nonzero curvature and torsion (and even non-metricity). By choosing an appropriate connection in this non-Riemannian manifold, the relaxed stress-free configurations fit together. This reference configuration now represents the initial arrangement of distributed dislocations and will be our starting point.

One possible way to model a crystalline solid with a large number of defects is to consider it in a continuum framework. Since the 1950s it has been appreciated that continuum mechanics of solids with distributed defects has a close connection with the differential geometry of manifolds with a Riemannian metric and torsion—a subject in mathematics that has found a wide range of applications in physics. For example, dislocation and disclination density tensors are closely related to torsion and curvature tensors, respectively, of a material connection. The geometric theory of dislocations has a long history. However, in spite of many efforts in the past few decades, a consistent systematic geometric continuum theory of solids with distributed defects, capable of calculating stress fields of defects and their evolution, is still missing. We should emphasize that the monograph of ZUBOV [86] and the work of ACHARYA [3] present stress calculations for distributed dislocations in nonlinear elastic solids, but are not geometric in the sense of the present paper.

KONDO [36] realized that in the presence of defects, the material manifold, which describes the stress-free state of a solid, is not necessarily Euclidean. He referred to the affine connection of this manifold as the material connection. Kondo also realized that the curvature of the material connection is a measure of the incompatibility of the material elements, and that the Bianchi identities are in some sense conservation equations for incompatibility. In [37], he considered a material manifold with an affine connection with nonzero curvature and torsion tensors, and discovered that torsion tensor is a measure of the density of dislocations. In these seminal papers, Kondo focused only on kinematic aspects; no stress calculations were presented. Independently, Bilby and his coworkers, in a series of papers [9–11] showed the relevance of non-Riemannian manifolds to solids with continuous distributions of dislocations. Although these seminal works made the crucial interpretation of dislocations as sources of torsion, none of them identified the geometric origin for the relevance of torsion. For example, for a solid with a single dislocation line, all the developments are intuitively based on the picture of a crystal with a single dislocation. Kröner and Seeger [38] and Kröner [39] used stress functions in a geometric framework in order to calculate stresses in a solid with distributed defects (see also [77]). None of these references provided any analytic solutions for stress fields of dislocations in nonlinear elastic solids. In this paper, for the first time, we calculate the stress fields of several examples of single and distributed dislocations in incompressible nonlinear elastic solids in a geometric framework. In particular, we show how an elastic solid with a single screw dislocation has a material manifold with a singular torsion distribution. By identifying the material manifold, the problem is then transformed to a standard nonlinear elasticity problem.

For the theory of evolution of defects, Kröner [40,43] proposed a field theory for dislocations, acknowledging that a Lagrangian formulation will ignore dissipation, which is present in the microscopic motion of dislocations. His theory involves a strain energy density $W = W(\mathbf{F}_e, \boldsymbol{\alpha})$, where $\mathbf{F}_e$ is the elastic part of the deformation gradient and $\boldsymbol{\alpha}$ is the dislocation density tensor. He argued that since strain energy density is a state variable, that is, independent of any history, it should depend explicitly only on quantities that are state variables. The tensor $\boldsymbol{\alpha}$ is a state variable because at any instant it can, in principle, be measured. Kröner [42] also associated a torque stress to dislocations. Le and Stumpf [45,46], building on ideas from [58] and [75], started with a "crystal connection" with nonzero torsion and zero curvature. They used the multiplicative decomposition of the deformation gradient $\mathbf{F} = \mathbf{F}_e\mathbf{F}_p$ and obtained some relations between torsion of the crystal connection and the elastic and plastic deformation gradients. They assumed that the free energy density is a function of $\mathbf{F}^e$ and its derivative with respect to the intermediate plastic configuration. Then they showed that material frame-indifference implies that the free energy density should explicitly depend on $\mathbf{F}_e$ and the push-forward of the crystal torsion to the intermediate configuration. Recently, in a series of papers, Acharya [3–5] presented a crystal plasticity theory that takes the dislocation density tensor as a primary internal variable without being explicitly present in the internal energy density. Berdichevsky [7] also presented a theory in which internal energy density explicitly depends on the dislocation density tensor.

These geometric ideas have also been presented in the physics literature. Katanaev and Volovich [35] started with equations of linear elasticity, and hence began their work outside the correct geometric realm of elasticity. They introduced a Lagrangian density for distributed dislocations and disclinations and assumed that it must be quadratic in both torsion and curvature tensors. They showed that the number of independent material constants can be reduced by assuming that there are displacement fields corresponding to the following three problems: (i) bodies with dislocations only, (ii) bodies with disclinations only, and (iii) bodies with no defects. MIRI and RIVIER [52] mentioned that extra matter is described geometrically as non-metricity of the material connection. RUGGIERO and TARTAGLIA [63] compared the Einstein–Cartan theory of gravitation to a geometric theory of defects in continua, and argued that in the linearized approximation, the equations describing defects can be interpreted as the Einstein–Cartan equations in three dimensions. However, similar to several other works in the physics literature, the early restriction to linearized approximation renders their approach non-geometric (see also [71] and [44]).

Einstein–Cartan gravity theory and defective solids. The Einstein–Cartan theory of gravity is a role model for a dynamical, geometric field theory of defect mechanics. This theory is a generalization of Einstein's general theory of relativity (GTR) involving torsion. Being inspired by the work of COSSERAT and COSSERAT [19] on generalized continua, that is, continua with microstructure, in the early 1920s Élie Cartan introduced a space–time with torsion before the discovery of spin. GTR treats spacetime as a possibly curved pseudo-Riemannian manifold. The connection on this manifold is taken to be the torsion-free Levi-Civita connection associated with a metric tensor. In general relativity, the geometry of spacetime, which is described by this metric tensor, is a dynamical variable, and its dynamics and coupling with matter are given by the Einstein equations [53], which relate the Ricci curvature tensor $R_{\mu\nu}$ to the energy-momentum tensor $T_{\mu\nu}$ in the following way (in suitable units):

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^{\alpha}_{\alpha} = T_{\mu\nu}. \tag{1.1}$$

It is very tempting to exploit similarities between GTR and a possible geometric theory of defects in solids: both theories describe the dynamics of the geometry of a curved space. However, the analogy falls short: in the case of defect mechanics, one needs to allow the material manifold to have torsion (for dislocations), which is nonexistent in GTR. A better starting point is Einstein–Cartan theory, which is a modified version of GTR that allows for torsion as well as for curvature [29]. In Einstein–Cartan theory, the metric determines the Levi-Civita part of the connection, and the other part (contorsion tensor), which is related to both torsion and metric, is a dynamical variable as well. The evolution of these variables is obtained by the field equations that are in a sense a generalization of Einstein's equations. These equations can be obtained from a variational principle just as in GTR. What one needs in the case of a continuum theory of solids with distributed dislocations is similar in spirit to Einstein–Cartan theory, and a possible approach to constructing such a theory is via an action that is compatible with the symmetries of the underlying physics [63]. However, there is an important distinction between Einstein–Cartan theory and dislocation mechanics: dissipation is a crucial ingredient in the mechanics of defects. In short, being a geometric field theory involving torsion and curvature, Einstein–Cartan theory serves as a valuable source of inspiration for our approach with all proper caveats taken into account.

A possible generalization can also be considered. The connection in Riemannian geometry is metric-compatible, and torsion-free. In Riemann–Cartan theory, the connection has torsion, but the metric is compatible. A further generalization can be made by allowing the connection to be non-metric-compatible. In this case, the covariant derivative of the metric becomes yet another dynamical variable [30]. This approach is relevant to defect mechanics as well, since non-metricity is believed to be related to point defects in solids [52]. However, in the present work we restrict ourselves to metric-compatible connections.

Dislocations and Torsion. As mentioned earlier, in a geometric formulation of anelasticity, dislocations are related to torsion. While it has been mentioned in many of the works cited above, the connection between the two concepts remains unclear. To shed light on this relation we focus on the simple case of a material manifold describing a single screw dislocation. Tod [69], in a paper on cosmological singularities, presented, the following family of 4-dimensional metrics

$$\mathrm{d}s^2 = -(\mathrm{d}t + \alpha \mathrm{d}\varphi)^2 + \mathrm{d}r^2 + \beta^2 r^2 \mathrm{d}\varphi^2 + (\mathrm{d}z + \gamma \mathrm{d}\varphi)^2, \tag{1.2}$$

which includes the special case ($\alpha = 0$, $\beta = 1$, $\mathrm{d}t = 0$), which can be interpreted as a screw dislocation parallel to the z-axis in three dimensions. Indeed, by considering the parallel transport of two vectors by infinitesimal amounts in each other's directions in this three-dimensional Riemannian manifold (apart from the z-axis), one can see that the z-axis contains a δ-function singularity of torsion. We will use the interpretation of this work in relativity in the context of dislocations in solids to make intuitively clear the relation between a single screw dislocation in a continuous medium and the torsion. Here, we use Tod's idea and show that his singular space–time restricted to three dimensions ($\alpha = 0$, $\beta = 1$) is the material manifold of a single screw dislocation. Using this material manifold we obtain the stress field when the dislocated body is an incompressible neo-Hookean solid in Section 6.

Here, a comment is in order. Since the 1950s, many researchers have worked on the connections between the mechanics of solids with distributed defects and non-Riemannian geometries. Unfortunately, most of these works focus on restatements of Kondo and Bilby's works and not on coupling mechanics with the geometry of defects. It is interesting that after more than six decades since the works of Kondo and Bilby there is not a single calculation of stress in a nonlinear elastic body with dislocations in a geometric framework. The present work introduces a geometric theory that can be used in nonlinear dislocation mechanics to calculate stresses. We show in several examples how one can use Riemann–Cartan geometry to calculate stresses in a dislocated body. We hope that these concrete examples demonstrate the power of geometric methods in generating new exact solutions in nonlinear anelasticity.

Major contributions of this paper. In this paper, we show that the mechanics of solids with distributed dislocations can be formulated as a nonlinear elasticity problem provided that the material manifold is chosen appropriately. Choosing a Weitzenböck manifold with a torsion tensor identified with a given dislocation density tensor, the body is stress free in the material manifold by construction. For classical nonlinear elastic solids, in order to calculate stresses one needs to know the changes of the relative distances, that is, a metric in the material manifold is needed. This metric is exactly the metric compatible with the Weitzenböck connection. We calculate the residual stress field of several distributed dislocations in incompressible nonlinear elastic solids. We use Cartan's moving frames to construct the appropriate material manifolds. Most of these exact solutions are new. Also, we discuss zero-stress dislocation distributions, present some non-trivial examples, and a covariant derivation of all the balance laws in a solid with distributed dislocations. The present work clearly shows the significance of geometric techniques in generating exact solutions in nonlinear dislocation mechanics. Application of our approach to distributed disclinations is presented in [85]. Extension of this geometric approach to distributed point defects will be the subject of future research.

This paper is structured as follows. In Section 2 we review Riemann–Cartan geometry. In particular, we discuss the familiar operations of Riemannian geometry for non-symmetric connections. We discuss bundle-valued differential forms and covariant exterior derivative, and then Cartan's moving frames. We also briefly comment on metrizability of non-symmetric connections. In Section 3, we critically review the classical dislocation mechanics, both linear and nonlinear. We critically reexamine existing definitions of the Burgers vector. Section 4 formulates dislocation mechanics in the language of Cartan's moving frames. The conditions under which a dislocation distribution is impotent (zero stress) are then discussed. Using Cartan's moving frames, we obtain some non-trivial zero-stress dislocation distributions. To the best of our knowledge, there is no previous result on zero-stress dislocations in the nonlinear setting in the literature. We also comment on linearization of the nonlinear theory. In Section 5 we derive the governing equations of a solid with distributed dislocations using energy balance and its covariance. Section 6 presents several examples of calculation of stresses induced by distributed dislocations in incompressible nonlinear elastic solids. We find the residual stresses for nonlinear elastic solids with no approximation or linearization. We start with a single screw dislocation and construct its material manifold. We then consider a parallel and cylindrically-symmetric distribution of screw dislocations. We calculate the residual stress field for an arbitrary distribution. We prove that for a distribution vanishing outside a finite-radius cylinder, stress distribution outside this cylinder depends only on the total Burgers vector and is identical to that of a single screw dislocation with the same Burgers vector.[1] As another example, we consider a uniformly and isotropically distributed screw dislocation and show that its material manifold is a three-sphere. Knowing that a three-sphere cannot be embedded into a three-dimensional Euclidean space, we conclude that there is no solution in the framework of classical nonlinear elasticity in the absence of couple stresses. This result holds for any nonlinear elastic solid, compressible or incompressible. Next, we consider an example of edge dislocations uniform in a collection of parallel planes but varying normal to the planes. Finally, we look at a radially-symmetric distribution of edge dislocations in two dimensions and calculate their residual stress field.

[1] This result is implicit in [3].

2. Riemann–Cartan Geometry

To establish notation we first review some facts about non-symmetric connections and the geometry of Riemann–Cartan manifolds. We then discuss bundle-valued differential forms and their intrinsic differentiation. Finally, we introduce Cartan's moving frames—a central tool in this paper. For more details see [13,24,31,32,55,56,64].

A linear (affine) connection on a manifold $\mathcal{B}$ is an operation $\nabla : \mathcal{X}(\mathcal{B}) \times \mathcal{X}(\mathcal{B}) \to \mathcal{X}(\mathcal{B})$, where $\mathcal{X}(\mathcal{B})$ is the set of vector fields on $\mathcal{B}$, such that $\forall\, f, f_1, f_2 \in C^\infty(\mathcal{B})$, $\forall\, a_1, a_2 \in \mathbb{R}$:

$$\text{i)}\ \nabla_{f_1\mathbf{X}_1+f_2\mathbf{X}_2}\mathbf{Y} = f_1\nabla_{\mathbf{X}_1}\mathbf{Y} + f_2\nabla_{\mathbf{X}_2}\mathbf{Y}, \tag{2.1}$$

$$\text{ii)}\ \nabla_{\mathbf{X}}(a_1\mathbf{Y}_1 + a_2\mathbf{Y}_2) = a_1\nabla_{\mathbf{X}}(\mathbf{Y}_1) + a_2\nabla_{\mathbf{X}}(\mathbf{Y}_2), \tag{2.2}$$

$$\text{iii)}\ \nabla_{\mathbf{X}}(f\mathbf{Y}) = f\nabla_{\mathbf{X}}\mathbf{Y} + (\mathbf{X}f)\mathbf{Y}. \tag{2.3}$$

$\nabla_{\mathbf{X}}\mathbf{Y}$ is called the covariant derivative of $\mathbf{Y}$ along $\mathbf{X}$. In a local chart $\{X^A\}$

$$\nabla_{\partial_A}\partial_B = \varGamma^C{}_{AB}\partial_C, \tag{2.4}$$

where $\varGamma^C{}_{AB}$ are Christoffel symbols of the connection and $\partial_A = \frac{\partial}{\partial x^A}$ are the natural bases for the tangent space corresponding to a coordinate chart $\{x^A\}$. A linear connection is said to be compatible with a metric $\mathbf{G}$ of the manifold if

$$\nabla_{\mathbf{X}} \langle\!\langle \mathbf{Y}, \mathbf{Z} \rangle\!\rangle_{\mathbf{G}} = \langle\!\langle \nabla_{\mathbf{X}}\mathbf{Y}, \mathbf{Z} \rangle\!\rangle_{\mathbf{G}} + \langle\!\langle \mathbf{Y}, \nabla_{\mathbf{X}}\mathbf{Z} \rangle\!\rangle_{\mathbf{G}}, \tag{2.5}$$

where $\langle\!\langle ., . \rangle\!\rangle_{\mathbf{G}}$ is the inner product induced by the metric $\mathbf{G}$. It can be shown that ∇ is compatible with $\mathbf{G}$ if and only if $\nabla\mathbf{G} = \mathbf{0}$, or in components

$$G_{AB|C} = \frac{\partial G_{AB}}{\partial X^C} - \varGamma^S{}_{CA}G_{SB} - \varGamma^S{}_{CB}G_{AS} = 0. \tag{2.6}$$

We consider an n-dimensional manifold $\mathcal{B}$ with the metric $\mathbf{G}$ and a $\mathbf{G}$-compatible connection ∇. Then $(\mathcal{B}, \nabla, \mathbf{G})$ is called a Riemann–Cartan manifold [14,25].

The torsion of a connection is a map $\boldsymbol{T} : \mathcal{X}(\mathcal{B}) \times \mathcal{X}(\mathcal{B}) \to \mathcal{X}(\mathcal{B})$ defined by

$$\boldsymbol{T}(\mathbf{X}, \mathbf{Y}) = \nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} - [\mathbf{X}, \mathbf{Y}]. \tag{2.7}$$

In components in a local chart $\{X^A\}$, $T^A{}_{BC} = \varGamma^A{}_{BC} - \varGamma^A{}_{CB}$. The connection is said to be symmetric if it is torsion-free, that is, $\nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} = [\mathbf{X}, \mathbf{Y}]$. It can be shown that on any Riemannian manifold $(\mathcal{B}, \mathbf{G})$ there is a unique linear connection ∇ that is compatible with $\mathbf{G}$ and is torsion-free [48]. Its Christoffel symbols are

$$\varGamma^C{}_{AB} = \frac{1}{2}G^{CD}\left(\frac{\partial G_{BD}}{\partial X^A} + \frac{\partial G_{AD}}{\partial X^B} - \frac{\partial G_{AB}}{\partial X^D}\right), \tag{2.8}$$

and the associated connection is the Levi-Civita connection. In a manifold with a connection, the Riemann curvature is a map $\boldsymbol{\mathcal{R}} : \mathcal{X}(\mathcal{B}) \times \mathcal{X}(\mathcal{B}) \times \mathcal{X}(\mathcal{B}) \to \mathcal{X}(\mathcal{B})$ defined by

$$\boldsymbol{\mathcal{R}}(\mathbf{X}, \mathbf{Y})\mathbf{Z} = \nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}\mathbf{Z} - \nabla_{\mathbf{Y}}\nabla_{\mathbf{X}}\mathbf{Z} - \nabla_{[\mathbf{X},\mathbf{Y}]}\mathbf{Z}, \tag{2.9}$$

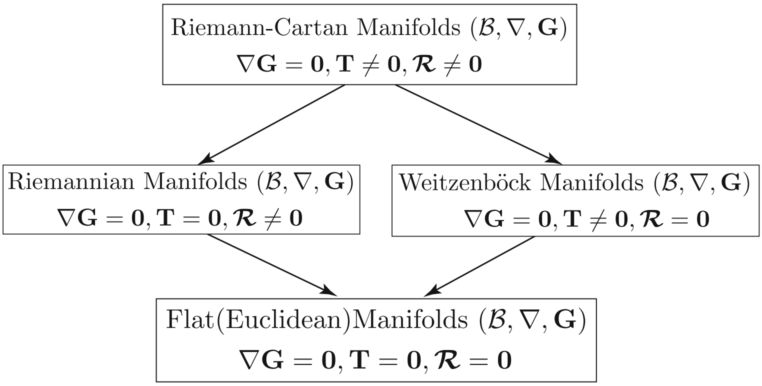

Fig. 1. Special cases of Riemann–Cartan manifolds

or, in components

$$\mathcal{R}^A{}_{BCD} = \frac{\partial \Gamma^A{}_{CD}}{\partial X^B} - \frac{\partial \Gamma^A{}_{BD}}{\partial X^C} + \Gamma^A{}_{BM}\Gamma^M{}_{CD} - \Gamma^A{}_{CM}\Gamma^M{}_{BD}. \quad (2.10)$$

A metric-affine manifold is a manifold equipped with both a connection and a metric: $(\mathcal{B}, \nabla, \mathbf{G})$. If the connection is metric compatible, the manifold is called a Riemann–Cartan manifold. If the connection is torsion free but has curvature $\mathcal{B}$ is called a Riemannian manifold. If the curvature of the connection vanishes but it has torsion $\mathcal{B}$ is called a Weitzenböck manifold. If both torsion and curvature vanish, $\mathcal{B}$ is a flat (Euclidean) manifold. Figure 1 schematically shows this classification. For a similar classification when the connection has non-metricity, that is, $\nabla\mathbf{G} \neq \mathbf{0}$ see [25].

The following are called Ricci formulas for vectors, one-forms, and $\binom{0}{2}$-tensors, respectively.

$$w^A{}_{|B|C} - w^A{}_{|C|B} = -\mathcal{R}^A{}_{BCM}w^M + T^M{}_{BC}w^A{}_{|M}, \quad (2.11)$$

$$\alpha_{A|B|C} - \alpha_{A|C|B} = \mathcal{R}^M{}_{BCA}\alpha_M + T^M{}_{BC}\alpha_{A|M}, \quad (2.12)$$

$$A_{AB|C|D} - A_{AB|D|C} = \mathcal{R}^M{}_{CDA}A_{MB} + \mathcal{R}^M{}_{CDB}A_{AM} + T^M{}_{CD}A_{AB|M}. \quad (2.13)$$

The Ricci curvature tensor is a $\binom{2}{0}$-tensor with the following coordinate representation: $R_{AB} = \mathcal{R}^C{}_{CAB}$. Scalar curvature is the trace of $\mathbf{R}$, that is, $\mathsf{R} = G^{AB}R_{AB}$. The Einstein tensor is defined as $\mathsf{E}_{AB} = R_{AB} - \frac{1}{2}\mathsf{R}G_{AB}$. In dimension three Ricci curvature (and equivalently the Einstein tensor) completely specifies the Riemann curvature tensor. Let us consider a 1-parameter family of metrics $G_{AB}(\varepsilon)$ such that

$$G_{AB}(0) = G_{AB}, \quad \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} G_{AB}(\varepsilon) = \delta G_{AB}, \quad (2.14)$$

δG_{AB} is called the metric variation. It can be shown that [17]

$$\delta G^{AB} = -G^{AM}G^{BN}\delta G_{MN}, \tag{2.15}$$

$$\delta\overline{\Gamma}^A{}_{BC} = \frac{1}{2}G^{AD}\left(\delta G_{CD|B} + \delta G_{BD|C} - \delta G_{BC|D}\right), \tag{2.16}$$

$$\delta\overline{R}_{AB} = \frac{1}{2}G^{MN}\big(\delta G_{AM|BN} + \delta G_{BN|AM} - \delta G_{AB|MN} - \delta G_{NM|BA}\big), \tag{2.17}$$

$$\delta\overline{\mathsf{R}} = -\Delta\delta G + G^{MP}G^{NQ}\delta G_{MN|PQ} - G^{AM}G^{BN}R_{AB}\delta G_{MN}, \tag{2.18}$$

where $\delta G = G^{AB}\delta G_{AB}$ and $\Delta = G^{AB}\nabla_A\nabla_B$.

Let us consider a metric connection $\Gamma^A{}_{BC}$ and its corresponding metric G_{AB} that is used in raising and lowering indices, for example $T_{BC}{}^A = G_{BM}G^{AN}T^M{}_{CN}$. Metric compatibility of the connection implies that $G_{AB|C} = G_{AB,C} - \Gamma^M{}_{CA}G_{MB} - \Gamma^M{}_{CB}G_{AM} = 0$. We use this identity to express $\Gamma^A{}_{BC}$ as

$$\Gamma^A{}_{BC} = \overline{\Gamma}^A{}_{BC} + K^A{}_{BC}, \tag{2.19}$$

where $\overline{\Gamma}^A{}_{BC}$ is the Levi-Civita connection of the metric and

$$K^A{}_{BC} = \frac{1}{2}\left(T^A{}_{BC} - T_{BC}{}^A - T_{CB}{}^A\right) = \frac{1}{2}\left(T^A{}_{BC} + T_B{}^A{}_C + T_C{}^A{}_B\right), \tag{2.20}$$

is called the contorsion tensor. Note that $\frac{1}{2}\left(\Gamma^A{}_{BC} + \Gamma^A{}_{CB}\right) = \overline{\Gamma}^A{}_{BC} + \left(T_B{}^A{}_C + T_C{}^A{}_B\right)$, that is, the symmetric part of the connection is not the Levi-Civita connection, in general. Similarly, the curvature tensor can be written in terms of curvature of the Levi-Civita connection and the contorsion tensor as

$$\mathcal{R}^A{}_{BCD} = \overline{\mathcal{R}}^A{}_{BCD} + K^A{}_{CD|B} - K^A{}_{BD|C} + K^A{}_{BM}K^M{}_{CD} - K^A{}_{CM}K^M{}_{BD}, \tag{2.21}$$

where the covariant derivatives of the contorsion tensor are with respect to the Levi-Civita connection. Finally, the Ricci tensor has the following relation with the Ricci tensor of the Levi-Civita connection

$$R_{AB} = \overline{R}_{AB} + K^M{}_{AB|M} - K^M{}_{MB|A} + K^N{}_{NM}K^M{}_{AB} - K^N{}_{AM}K^M{}_{NB}. \tag{2.22}$$

The Levi-Civita tensor ε_{ABC} is defined as $\varepsilon_{ABC} = \sqrt{G}\ \epsilon_{ABC}$, where $G = \det \mathbf{G}$ and

$$\epsilon_{ABC} = \begin{cases} 1 & (ABC) \text{ is an even permutation of } (123), \\ -1 & (ABC) \text{ is an odd permutation of } (123), \\ 0 & \text{otherwise}, \end{cases} \tag{2.23}$$

is the Levi-Civita symbol. For a $\mathbf{G}$-compatible connection, $\varepsilon_{ABC|D} = 0$.

2.1. Bundle-Valued Differential Forms

Here we review some definitions and operations on vector-valued and covector-valued differential forms, that is, differential forms that take values in a vector bundle rather than in $\mathbb{R}$ (torsion form is an example of a vector-valued 2-form). See also [50, Chapter 16] for a more detailed discussion of operations on vector bundles. A more accessible presentation can be found in [22, Chapter 9]. Other treatments of vector-valued forms can be seen in [15] and [23]. A shorter version of what follows was presented in [34].

We consider an n-dimensional Riemann–Cartan manifold $(\mathcal{B}, \nabla, \mathbf{G})$. For the sake of clarity, we consider mainly 2-tensors on $\mathcal{B}$. However, it is straightforward to extend all the concepts presented here to tensors of arbitrary order. Consider a covariant 2-tensor $\mathbf{T} \in T_2^0(\mathcal{B})$. Its Hodge star with respect to the second argument is defined as

$$*_2 : T_2^0(\mathcal{B}) \to \Omega^1(\mathcal{B}) \otimes \Omega^{n-1}(\mathcal{B}); \quad \mathbf{T} \mapsto *_2\mathbf{T}, \tag{2.24}$$

such that $\forall\, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_n \in T\mathcal{B}$

$$(*_2\mathbf{T})_X(\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n) = \big(*\mathbf{T}(\boldsymbol{u}_1, \cdot)\big)_X(\boldsymbol{u}_2, \ldots, \boldsymbol{u}_n) \quad \forall X \in \mathcal{B}, \tag{2.25}$$

where $*$ is the standard Hodge star operator. Clearly, $*_2\mathbf{T}$ is in $\Omega^1(\mathcal{B}) \otimes \Omega^{n-1}(\mathcal{B})$, that is, an element of $T_n^0(\mathcal{B})$ antisymmetric in the last $n-1$ arguments. In coordinate notation, if we write $\mathbf{T} = T_{AB}\, \mathrm{d}X^A \otimes \mathrm{d}X^B$, then $*_2\mathbf{T} = T_{AB}\, \mathrm{d}X^A \otimes *\mathrm{d}X^B$. Now, define the area-forms $\omega_A := (-1)^{A-1}\mathrm{d}X^1 \wedge \ldots \wedge \widehat{\mathrm{d}X^A} \wedge \ldots \wedge \mathrm{d}X^n$, where the hat means that $\mathrm{d}X^A$ is omitted. It is clear that $\{\omega_A\}$ is a basis for $\Omega^{n-1}(\mathcal{B})$, and hence $\{\mathrm{d}X^A \otimes \omega_B\}$ is a basis for $\Omega^1(\mathcal{B}) \otimes \Omega^{n-1}(\mathcal{B})$. One can check that the components of the tensor $*_2\mathbf{T}$ in this basis are

$$|\det \mathbf{G}|^{1/2} T_{AC} G^{CB} = |\det \mathbf{G}|^{1/2} T_A{}^B. \tag{2.26}$$

Finally, note that (2.24) can be easily extended to contravariant and mixed 2−tensors by simply lowering the second index. For instance, if $\mathbf{S} \in T_0^2(\mathcal{B})$, then we define $*_2\mathbf{S} \in T\mathcal{B} \otimes \Omega^{n-1}(\mathcal{B})$ such that, $\forall\, \alpha \in T^*\mathcal{B}$ and $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n \in T\mathcal{B}$, one has:

$$(*_2\mathbf{S})_X(\alpha, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n) = \big(*(\mathbf{S}(\alpha, \cdot))^\flat\big)_X(\boldsymbol{u}_2, \ldots, \boldsymbol{u}_n) \quad \forall\, X \in \mathcal{B}. \tag{2.27}$$

Recall that the flat $(\cdot)^\flat$ and sharp $(\cdot)^\sharp$ operations refer to lowering and raising the indices using the metric $\mathbf{G}$, that is, $^\flat : T\mathcal{B} \to T^*\mathcal{B}$ and $^\sharp : T^*\mathcal{B} \to T\mathcal{B}$. Let $\beta \in \Omega^1(\mathcal{B})$. We have:

$$*\,\beta = \langle \beta^\sharp, \mu \rangle, \tag{2.28}$$

where μ is the $\boldsymbol{G}$-volume form. This result is analogous to the well-known relation $*X^\flat = i_X\mu$, where $X \in T\mathcal{B}$ and i denotes the contraction operation, see [1]. As a corollary, for $\mathbf{T} \in T_2^0(\mathcal{B})$, one has

$$*_2\,\mathbf{T} = \langle \mathbf{T}^{\sharp_2}, \mu \rangle, \tag{2.29}$$

where $\sharp_2$ denotes the operator of raising the second index. Now let $S = \partial V \subset \mathcal{B}$ be an $(n-1)-$surface with Riemannian area-form ν and consistently oriented unit normal vector field $\boldsymbol{N}$ and $\mathbf{T} \in T^0_2(\mathcal{B})$, then

$$\int_S \mathbf{T}(\mathbf{v}, \boldsymbol{N})\nu = \int_S \langle \mathbf{v}, *_2\mathbf{T}\rangle, \quad \forall \mathbf{v} \in T\mathcal{B}. \tag{2.30}$$

The proof follows by noting that $\int_S \mathbf{T}(\mathbf{v}, \boldsymbol{N})\nu = \int_S \langle \mathbf{T}(\mathbf{v}, \cdot), \boldsymbol{N}\rangle\nu$, then recalling that, for a one-form β, $\int_S \langle \beta, \boldsymbol{N}\rangle\, \nu = \int_S \langle \beta^\sharp, \mu\rangle$ and, finally, appealing to (2.29).

We now define two types of products, namely, an inner-exterior and an outer-exterior product that we denote by $\dot{\wedge}$ and $\overset{\otimes}{\wedge}$, respectively. Let us first define the $\dot{\wedge}$-product

$$\dot{\wedge} : \Big(T\mathcal{B} \otimes \Omega^1(\mathcal{B})\Big) \times \Big(T^*\mathcal{B} \otimes \Omega^{n-1}(\mathcal{B})\Big) \longrightarrow \Omega^n(\mathcal{B}); \quad (\mathbf{T}, \mathbf{S}) \longmapsto \mathbf{T}\dot{\wedge}\mathbf{S}, \tag{2.31}$$

such that, for all $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in T\mathcal{B}$, one has

$$(\mathbf{T}\dot{\wedge}\mathbf{S})_X(\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n) = \sum (\text{sign }\tau)\langle \mathbf{T}_X(\cdot, \boldsymbol{v}_{\tau(1)}), \mathbf{S}_X(\cdot, \boldsymbol{v}_{\tau(2)}, \ldots, \boldsymbol{v}_{\tau(n)})\rangle, \tag{2.32}$$

$\forall X \in \mathcal{B}$, where the sum is over all the $(1, n-1)$ shuffles. This product can be defined for any arbitrary order $k \leqq n$ as well as on $\big(T^*\mathcal{B} \otimes \Omega^1(\mathcal{B})\big) \times \big(T\mathcal{B} \otimes \Omega^{k-1}(\mathcal{B})\big)$. Note that the $\dot{\wedge}$-product is simply a contraction on the first index and a wedge product on the other indices. For example, if one takes $\boldsymbol{u} \otimes \alpha \in T\mathcal{B} \otimes \Omega^1(\mathcal{B})$, and $\beta \otimes \omega \in T^*\mathcal{B} \otimes \Omega^{n-1}(\mathcal{B})$, one has

$$(\boldsymbol{u} \otimes \alpha)\dot{\wedge}(\beta \otimes \omega) = \langle \boldsymbol{u}, \beta\rangle \alpha \wedge \omega, \tag{2.33}$$

where $\alpha \wedge \omega$ defines a volume-form provided it is not degenerate. To this end, one can readily verify that, for $\mathbf{T} \in T^2_0(\mathcal{B})$ and $\mathbf{S} \in T^0_2(\mathcal{B})$, one can write

$$\mathbf{S} \,\dot{\wedge}\, (*_2\mathbf{T}) = (\mathbf{S} : \mathbf{T})\, \mu. \tag{2.34}$$

Further, one can also show that, for $\mathbf{T} \in T^2_0(\mathcal{B})$ (analogous results hold for any tensor type), one has

$$\mathbf{T}(\alpha, \beta)\, \mu = (-1)^{n-1}\langle \alpha, *_2\mathbf{T}\rangle \wedge \beta = (\alpha \otimes \beta)\dot{\wedge}(*_2\mathbf{T}), \tag{2.35}$$

for all $\alpha, \beta \in T^*\mathcal{B}$. The proof follows directly from the definition of the Hodge star as in [1]. We now define the $\overset{\otimes}{\wedge}$-product

$$\overset{\otimes}{\wedge}: \big(T\mathcal{B} \otimes \Omega^1(\mathcal{B})\big) \times \big(T\mathcal{B} \otimes \Omega^{n-1}(\mathcal{B})\big) \longrightarrow T\mathcal{B} \otimes \Omega^n(\mathcal{B}); \quad (\mathbf{T}, \mathbf{S}) \longmapsto \mathbf{T}, \overset{\otimes}{\wedge} \mathbf{S}, \tag{2.36}$$

such that, $\forall\ \boldsymbol{v}_1, \ldots \boldsymbol{v}_n \in T\mathcal{B}$, one has

$$(\mathbf{T} \overset{\otimes}{\wedge} \boldsymbol{\omega})_X(\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n) = \sum (\text{sign }\tau)\mathbf{T}_X(\cdot, \boldsymbol{v}_{\tau(1)}) \otimes \mathbf{S}_X(\cdot, \boldsymbol{v}_{\tau(2)}, \ldots, \boldsymbol{v}_{\tau(n)}), \tag{2.37}$$

$\forall X \in \mathcal{B}$, where the sum is over all the $(1, n-1)$ shuffles. An analogous product can be defined on $(T^*\mathcal{B} \otimes T^*\mathcal{B}) \times \Omega^{n-1}(\mathcal{B})$. The $\overset{\otimes}{\wedge}$-product is simply a tensor product on the first index and a wedge product on the other indices.

Differentiation of Bundle-Valued Forms. We now proceed to define a differentiation $\mathscr{D}$ on vector and covector-valued $(k-1)$-forms. The differentiation $\mathscr{D}$ combines the exterior derivative d, which has a topological character, with the covariant derivative ∇ with respect to the affine connection, which has a metric character if the connection is metric compatible. To this end, recall that, in components, the covariant derivative $\nabla \boldsymbol{v}$ of a vector field $\boldsymbol{v} = v^A \mathbf{e}_A$ on $T\mathcal{B}$ is given by $\nabla_B v^A = v^A{}_{|B} = \partial v^A / \partial X^B + \Gamma^A{}_{BC} v^C$, where $\Gamma^A{}_{BC}$ are the connection coefficients. This suggests that $\nabla \boldsymbol{v}$ can be expressed as a mixed 2-tensor, that is, a vector-valued one-form $\nabla \boldsymbol{v} = v^A{}_{|B} \mathbf{e}_A \otimes \mathrm{d}X^B$. In particular, one has $\nabla \mathbf{e}_B = \mathbf{e}_A \otimes \Gamma^A{}_{CB} \mathrm{d}X^C = \mathbf{e}_A \otimes \omega^A{}_B$, where $\omega^A{}_B = \Gamma^A{}_{CB} \mathrm{d}X^C$ are called the connection one-forms. Let $\mathbb{F}$ denote either $T\mathcal{B}$ or $T^*\mathcal{B}$, and let k be any integer $\leqq n$. We define the differential operator

$$\mathscr{D} : \mathbb{F} \otimes \Omega^{k-1}(\mathcal{B}) \longrightarrow \mathbb{F} \otimes \Omega^k(\mathcal{B}); \quad \boldsymbol{\mathcal{T}} \longmapsto \mathscr{D}\boldsymbol{\mathcal{T}}, \tag{2.38}$$

by

$$\langle \boldsymbol{u}, \mathscr{D}\boldsymbol{\mathcal{T}} \rangle = \mathrm{d}(\langle \boldsymbol{u}, \boldsymbol{\mathcal{T}} \rangle) - \nabla \boldsymbol{u} \dot{\wedge} \boldsymbol{\mathcal{T}}, \quad \forall\, \boldsymbol{u} \in \mathbb{F}^*, \tag{2.39}$$

where d is the regular exterior derivative of forms and ∇ is the covariant derivative of tensors. Note that for $k = 0$, $\mathscr{D}$ reduces to the regular covariant derivative, while for $k = n$, $\mathscr{D}$ is identically zero.

Remark 2.1. In order for (2.39) to provide a valid definition of $\mathscr{D}$, one needs to show that its right-hand side depends only on the point values of $\boldsymbol{u}$ and, hence, uniquely defines the differential $\mathscr{D}\boldsymbol{\mathcal{T}}$. Note that for any function $f \in \Omega^0(\mathcal{B})$, one has

$$\mathrm{d}\langle f\, \boldsymbol{u}, \boldsymbol{\mathcal{T}} \rangle = d(f \wedge \langle \boldsymbol{u}, \boldsymbol{\mathcal{T}} \rangle) = \mathrm{d}f \wedge \langle \boldsymbol{u}, \boldsymbol{\mathcal{T}} \rangle + f \mathrm{d}(\langle \boldsymbol{u}, \boldsymbol{\mathcal{T}} \rangle). \tag{2.40}$$

On the other hand, one can easily verify that

$$\nabla(f\boldsymbol{u}) \dot{\wedge} \boldsymbol{\mathcal{T}} = (\boldsymbol{u} \otimes \mathrm{d}f) \dot{\wedge} \boldsymbol{\mathcal{T}} + f \nabla \boldsymbol{u} \dot{\wedge} \boldsymbol{\mathcal{T}} = \mathrm{d}f \wedge \langle \boldsymbol{u}, \boldsymbol{\mathcal{T}} \rangle + f \nabla \boldsymbol{u} \dot{\wedge} \boldsymbol{\mathcal{T}}, \tag{2.41}$$

which proves the claim.

Alternatively, the differential operator $\mathscr{D}$ can be defined by its action on elements of $\mathbb{F} \otimes \Omega^{k-1}(\mathcal{B})$ of the type $\alpha \otimes \omega$, where $\alpha \in \mathbb{F},\ \omega \in \Omega^{k-1}(\mathcal{B})$:

$$\mathscr{D}(\alpha \otimes \omega) = \nabla \alpha \overset{\otimes}{\wedge} \omega + \alpha \otimes \mathrm{d}\omega, \tag{2.42}$$

and extending it to $\mathbb{F} \otimes \Omega^{n-1}(\mathcal{B})$ by linearity. To prove this statement, one only needs to check that (2.42) is equivalent to the definition in (2.39). Given $\boldsymbol{u} \in \mathbb{F}^*$, (2.39) reads as

$$\boldsymbol{u} \cdot \mathscr{D}(\alpha \otimes \omega) = d\left((\boldsymbol{u} \cdot \alpha)\, \omega\right) - \nabla \boldsymbol{u}(\alpha, \cdot) \wedge \omega. \tag{2.43}$$

Now, note that $d(\boldsymbol{u}\cdot\alpha) = \nabla(\boldsymbol{u}\cdot\alpha) = \nabla\alpha(\boldsymbol{u},\cdot) + \nabla\boldsymbol{u}(\alpha,\cdot)$ (by definition of ∇) in order to get

$$\boldsymbol{u}\cdot\mathscr{D}(\alpha\otimes\omega) = (\boldsymbol{u}\cdot\alpha)\,\mathrm{d}\omega + (\nabla\alpha(\boldsymbol{u},\cdot) + \nabla\boldsymbol{u}(\alpha,\cdot))\wedge\omega - \nabla\boldsymbol{u}(\alpha,\cdot)\wedge\omega$$
$$= (\boldsymbol{u}\cdot\alpha)\,\mathrm{d}\omega + \nabla\alpha(\boldsymbol{u},\cdot)\wedge\omega. \qquad (2.44)$$

The differential operator $\mathscr{D}$ is identical to *Cartan's exterior covariant differential* (reviewed in [22, Chapter 9], see also [74]) as we will show later on.

For a $\binom{2}{0}$-tensor $\mathbf{T}$, we have

$$(\mathrm{Div}\mathbf{T})\otimes\mu = \mathscr{D}(*_2\mathbf{T}^{\flat_2}). \qquad (2.45)$$

To show this, given that $\mathrm{Div}(\langle\alpha,\mathbf{T}\rangle) = \langle\alpha,\mathrm{Div}\mathbf{T}\rangle + \nabla\alpha : \mathbf{T}$ and appealing to the divergence theorem, we obtain the following identity

$$\int_V \langle\alpha,\mathrm{Div}\mathbf{T}\rangle\,\mu = \int_{\partial V}\mathbf{T}(\alpha,\boldsymbol{N}^\flat)\nu - \int_V(\nabla\alpha:\mathbf{T})\,\mu,\quad \forall\,\alpha\in T^*(\mathcal{B}), \qquad (2.46)$$

for any open subset $V\subset\mathcal{B}$. It follows from (2.30) and Stokes' theorem that

$$\int_{\partial V}\mathbf{T}(\alpha,\boldsymbol{N}^\flat)\,\nu = \int_{\partial V}\left\langle\alpha,*_2\mathbf{T}^{\flat_2}\right\rangle = \int_V \mathrm{d}(\langle\alpha,*_2\mathbf{T}^{\flat_2}\rangle). \qquad (2.47)$$

Further, from (2.34) and the definition of $\mathscr{D}$, we have

$$\int_V\langle\alpha,\mathrm{Div}\mathbf{T}\rangle\mu = \int_V\left[\mathrm{d}(\langle\alpha,*_2\mathbf{T}^{\flat_2}\rangle) - \nabla\alpha\dot{\wedge}(*_2\mathbf{T}^{\flat_2})\right]$$
$$= \int_V\langle\alpha,\mathscr{D}(*_2\mathbf{T}^{\flat_2})\rangle, \qquad (2.48)$$

for all $\alpha\in T^*(\mathcal{B})$ and for any open subset $V\subset\mathcal{B}$, which concludes the proof of (2.45).

2.2. *Cartan's Moving Frames*

Let us consider a frame field $\{\mathbf{e}_\alpha\}_{\alpha=1}^n$ which at every point of an n-dimensional manifold $\mathcal{B}$ forms a basis for the tangent space. We assume that this frame is orthonormal, that is, $\langle\!\langle\mathbf{e}_\alpha,\mathbf{e}_\beta\rangle\!\rangle_{\mathbf{G}} = \delta_{\alpha\beta}$. This is, in general, a non-coordinate basis for the tangent space. Given a coordinate basis $\{\partial_A\}$ an arbitrary frame field $\{\mathbf{e}_\alpha\}$ is obtained by a $GL(N,\mathbb{R})$-rotation of $\{\partial_A\}$ as $\mathbf{e}_\alpha = \mathsf{F}_\alpha{}^A\partial_A$, such that orientation is preserved, that is, $\det\mathsf{F}_\alpha{}^A > 0$. We know that for the coordinate frame $[\partial_A,\partial_B] = 0$, but for the non-coordinate frame field we have $[\mathbf{e}_\alpha,\mathbf{e}_\alpha] = -c^\gamma{}_{\alpha\beta}\mathbf{e}_\gamma$, where $c^\gamma{}_{\alpha\beta}$ are components of the object of anhonolomy. Note that for scalar fields f, g and vector fields $\mathbf{X}$, $\mathbf{Y}$ on $\mathcal{B}$ we have

$$[f\mathbf{X},g\mathbf{Y}] = fg[\mathbf{X},\mathbf{Y}] + f\,(\mathbf{X}[g])\,\mathbf{Y} - g\,(\mathbf{Y}[f])\,\mathbf{X}. \qquad (2.49)$$

Therefore

$$c^\gamma{}_{\alpha\beta} = \mathsf{F}_\alpha{}^A\mathsf{F}_\beta{}^B\left(\partial_A\mathsf{F}^\gamma{}_B - \partial_B\mathsf{F}^\gamma{}_A\right), \qquad (2.50)$$

where $\mathsf{F}^{\gamma}{}_{B}$ is the inverse of $\mathsf{F}_{\gamma}{}^{B}$. The frame field $\{\mathbf{e}_{\alpha}\}$ defines the coframe field $\{\vartheta^{\alpha}\}_{\alpha=1}^{n}$ such that $\vartheta^{\alpha}(\mathbf{e}_{\beta}) = \delta^{\alpha}_{\beta}$. The object of anholonomy is defined as $c^{\gamma} = \mathrm{d}\vartheta^{\gamma}$. Writing this in the coordinate basis we have

$$
\begin{aligned}
c^{\gamma} &= d\left(\mathsf{F}^{\gamma}{}_{B}\mathrm{d}X^{B}\right) \\
&= \partial_{A}\mathsf{F}^{\gamma}{}_{B}\mathrm{d}X^{A}\wedge\mathrm{d}X^{B} \\
&= \sum_{A<B}\left(\partial_{A}\mathsf{F}^{\gamma}{}_{B}-\partial_{B}\mathsf{F}^{\gamma}{}_{A}\right)\mathrm{d}X^{A}\wedge\mathrm{d}X^{B} \\
&= \sum_{\alpha<\beta}\mathsf{F}_{\alpha}{}^{A}\mathsf{F}_{\beta}{}^{B}\left(\partial_{A}\mathsf{F}^{\gamma}{}_{B}-\partial_{B}\mathsf{F}^{\gamma}{}_{A}\right)\vartheta^{\alpha}\wedge\vartheta^{\beta} \\
&= \sum_{\alpha<\beta}c^{\gamma}{}_{\alpha\beta}\vartheta^{\alpha}\wedge\vartheta^{\beta} \\
&= c^{\gamma}{}_{\alpha\beta}(\vartheta^{\alpha}\wedge\vartheta^{\beta}),
\end{aligned}
\tag{2.51}
$$

where $\{(\vartheta^{\alpha}\wedge\vartheta^{\beta})\} = \{\vartheta^{\alpha}\wedge\vartheta^{\beta}\}_{\alpha<\beta}$ is a basis for 2-forms.

Connection 1-forms are defined as $\nabla\mathbf{e}_{\alpha} = \mathbf{e}_{\gamma}\otimes\omega^{\gamma}{}_{\alpha}$. The corresponding connection coefficients are defined as

$$
\nabla_{\mathbf{e}_{\beta}}\mathbf{e}_{\alpha} = \langle\omega^{\gamma}{}_{\alpha},\mathbf{e}_{\beta}\rangle\,\mathbf{e}_{\gamma} = \omega^{\gamma}{}_{\beta\alpha}\mathbf{e}_{\gamma}. \tag{2.52}
$$

In other words, $\omega^{\gamma}{}_{\alpha} = \omega^{\gamma}{}_{\beta\alpha}\vartheta^{\beta}$. Similarly, $\nabla\vartheta^{\alpha} = -\omega^{\alpha}{}_{\gamma}\vartheta^{\gamma}$, and

$$
\nabla_{\mathbf{e}_{\beta}}\vartheta^{\alpha} = -\omega^{\alpha}{}_{\beta\gamma}\vartheta^{\gamma}. \tag{2.53}
$$

The relation between the connection coefficients in the two coordinate systems is

$$
\omega^{\gamma}{}_{\alpha\beta} = \mathsf{F}_{\alpha}{}^{A}\mathsf{F}_{\beta}{}^{B}\mathsf{F}^{\gamma}{}_{C}\Gamma^{C}{}_{AB} - \mathsf{F}_{\alpha}{}^{A}\mathsf{F}_{\beta}{}^{B}\partial_{A}\mathsf{F}^{\gamma}{}_{B}. \tag{2.54}
$$

And equivalently

$$
\Gamma^{A}{}_{BC} = \mathsf{F}^{\beta}{}_{B}\mathsf{F}^{\gamma}{}_{C}\mathsf{F}_{\alpha}{}^{A}\omega^{\alpha}{}_{\beta\gamma} + \mathsf{F}_{\alpha}{}^{A}\partial_{B}\mathsf{F}^{\alpha}{}_{C}. \tag{2.55}
$$

In the non-coordinate basis, torsion has the following components

$$
T^{\alpha}{}_{\beta\gamma} = \omega^{\alpha}{}_{\beta\gamma} - \omega^{\alpha}{}_{\gamma\beta} + c^{\alpha}{}_{\beta\gamma}. \tag{2.56}
$$

Similarly, the curvature tensor has the following components with respect to the frame field

$$
\mathcal{R}^{\alpha}{}_{\beta\lambda\mu} = \partial_{\beta}\omega^{\alpha}{}_{\lambda\mu} - \partial_{\lambda}\omega^{\alpha}{}_{\beta\mu} + \omega^{\alpha}{}_{\beta\xi}\omega^{\xi}{}_{\lambda\mu} - \omega^{\alpha}{}_{\lambda\xi}\omega^{\xi}{}_{\beta\mu} + \omega^{\alpha}{}_{\xi\mu}c^{\xi}{}_{\beta\lambda}. \tag{2.57}
$$

In the orthonormal frame $\{\mathbf{e}_{\alpha}\}$, the metric tensor has the simple representation $\mathbf{G} = \delta_{\alpha\beta}\vartheta^{\alpha}\otimes\vartheta^{\beta}$. Assuming that the connection ∇ is metric compatible, that is, $\nabla\mathbf{G} = \mathbf{0}$, we obtain the following metric compatibility constraints on the connection 1-forms:

$$
\delta_{\alpha\gamma}\omega^{\gamma}{}_{\beta} + \delta_{\beta\gamma}\omega^{\gamma}{}_{\alpha} = 0. \tag{2.58}
$$

Torsion and curvature 2-forms are defined as

$$\mathcal{T}^\alpha = \mathrm{d}\vartheta^\alpha + \omega^\alpha{}_\beta \wedge \vartheta^\beta, \tag{2.59}$$

$$\mathcal{R}^\alpha{}_\beta = \mathrm{d}\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma \wedge \omega^\gamma{}_\beta, \tag{2.60}$$

where d is the exterior derivative. These are called *Cartan's structural equations*. Torsion two-form is written as $\boldsymbol{\mathcal{T}} = \mathbf{e}_\alpha \otimes \mathcal{T}^\alpha = \partial_A \otimes T^A$, where $\mathcal{T}^\alpha = \mathsf{F}^\alpha{}_A T^A$. Bianchi identities read:

$$D\mathcal{T}^\alpha := \mathrm{d}\mathcal{T}^\alpha + \omega^\alpha{}_\beta \wedge \mathcal{T}^\beta = \mathcal{R}^\alpha{}_\beta \wedge \vartheta^\beta, \tag{2.61}$$

$$D\mathcal{R}^\alpha{}_\beta := \mathrm{d}\mathcal{R}^\alpha{}_\beta + \omega^\alpha{}_\gamma \wedge \mathcal{R}^\gamma{}_\beta - \omega^\gamma{}_\beta \wedge \mathcal{R}^\alpha{}_\gamma = 0, \tag{2.62}$$

where D is the covariant exterior derivative. Note that for a flat manifold $D\mathcal{T}^\alpha = 0$. We now show that D is identical to $\mathscr{D}$. For the sake of concreteness, we show this for a vector-valued 2-form. Since $\boldsymbol{\mathcal{T}}$ is a vector-valued 2-form we consider an arbitrary 1-form $\boldsymbol{\eta}$. From (2.39) we have

$$\langle \boldsymbol{\eta}, \mathscr{D}\boldsymbol{\mathcal{T}} \rangle = \mathrm{d}(\langle \boldsymbol{\eta}, \boldsymbol{\mathcal{T}} \rangle) - \nabla\boldsymbol{\eta} \dot{\wedge} \boldsymbol{\mathcal{T}}. \tag{2.63}$$

Note that $\boldsymbol{\mathcal{T}} = \mathbf{e}_\alpha \otimes \mathcal{T}^\alpha$ and let us take $\boldsymbol{\eta}$ to be ϑ^α. Thus

$$\begin{aligned}
\langle \vartheta^\alpha, \mathscr{D}\boldsymbol{\mathcal{T}} \rangle &= \mathrm{d}(\langle \vartheta^\alpha, \mathbf{e}_\beta \otimes \mathcal{T}^\beta \rangle) - \nabla\vartheta^\alpha \dot{\wedge} (\mathbf{e}_\beta \otimes \mathcal{T}^\beta) \\
&= \mathrm{d}\mathcal{T}^\alpha + \omega^\alpha{}_\gamma \vartheta^\gamma \dot{\wedge} (\mathbf{e}_\beta \otimes \mathcal{T}^\beta) \\
&= \mathrm{d}\mathcal{T}^\alpha + \omega^\alpha{}_\gamma \wedge \mathcal{T}^\beta \langle \vartheta^\gamma, \mathbf{e}_\beta \rangle \\
&= \mathrm{d}\mathcal{T}^\alpha + \omega^\alpha{}_\beta \wedge \mathcal{T}^\beta. \quad \square
\end{aligned} \tag{2.64}$$

From here on we use the symbol D for covariant exterior derivative.

Suppose a frame field $\{\mathbf{e}_\alpha\}$ is given. Then one may be interested in a connection ∇ such that in $(\mathcal{B}, \nabla)$ the frame field is parallel everywhere. This means that $\nabla\mathbf{e}_\alpha = \omega^\beta{}_\alpha \mathbf{e}_\beta = \mathbf{0}$, that is, the connection 1-forms vanish with respect to the frame field or $\omega^\beta{}_{\gamma\alpha} = 0$. Using this and (2.54) we have the following connection coefficients in the coordinate frame

$$\varGamma^C{}_{AB} = \mathsf{F}_\alpha{}^C \partial_A \mathsf{F}^\alpha{}_B. \tag{2.65}$$

This is called the Weitzenböck connection [21,76]. Its torsion reads

$$T^C{}_{AB} = \mathsf{F}_\alpha{}^C \left(\partial_A \mathsf{F}^\alpha{}_B - \partial_B \mathsf{F}^\alpha{}_A \right). \tag{2.66}$$

Let us denote the Levi-Civita connection 1-form by $\overline{\omega}^\alpha{}_\beta$. Distortion 1-form is defined as $N^\alpha{}_\beta = \omega^\alpha{}_\beta - \overline{\omega}^\alpha{}_\beta$. Thus, torsion one-form can be written as

$$\begin{aligned}
\mathcal{T}^\alpha &= \mathrm{d}\vartheta^\alpha + \left(\overline{\omega}^\alpha{}_\beta + N^\alpha{}_\beta\right) \wedge \vartheta^\beta \\
&= \left(\mathrm{d}\vartheta^\alpha + \overline{\omega}^\alpha{}_\beta \wedge \vartheta^\beta\right) + N^\alpha{}_\beta \wedge \vartheta^\beta \\
&= N^\alpha{}_\beta \wedge \vartheta^\beta,
\end{aligned} \tag{2.67}$$

where we used the fact that torsion of the Levi-Civita connection vanishes. Curvature 2-form has the following relation with the Levi-Civita curvature 2-form:

$$\mathcal{R}^\alpha{}_\beta = \overline{\mathcal{R}}^\alpha{}_\beta + \overline{D} N^\alpha{}_\beta + N^\alpha{}_\gamma \wedge N^\gamma{}_\beta, \tag{2.68}$$

where $\overline{D}$ is the covariant exterior derivative with respect to the Levi-Civita connection forms.

Example 2.2. Let (R, Θ, Z), $R \geqq 0, 0 \leqq \Theta < 2\pi$, $Z \in \mathbb{R}$, denote the cylindrical coordinates in Euclidean space. The metric is given by

$$\boldsymbol{G} = \mathrm{d}R \otimes \mathrm{d}R + R^2 \mathrm{d}\Theta \otimes \mathrm{d}\Theta + \mathrm{d}Z \otimes \mathrm{d}Z. \tag{2.69}$$

We choose the following orthonormal non-coordinate coframes: $\vartheta^1 = \mathrm{d}R$, $\vartheta^2 = R\mathrm{d}\Theta$, $\vartheta^3 = \mathrm{d}Z$. Metric compatibility of the connection implies that there are only three non-zero connection one-forms. The matrix of connection one-forms has the following form:

$$\boldsymbol{\omega} = [\omega^\alpha{}_\beta] = \begin{pmatrix} 0 & \omega^1{}_2 & -\omega^3{}_1 \\ -\omega^1{}_2 & 0 & \omega^2{}_3 \\ \omega^3{}_1 & -\omega^2{}_3 & 0 \end{pmatrix}. \tag{2.70}$$

Using Cartan's first structural equations ($0 = \mathrm{d}\vartheta^\alpha + \omega^\alpha{}_\beta \wedge \vartheta^\beta$) we find the Levi-Civita connection 1-forms:

$$\boldsymbol{\omega} = \begin{pmatrix} 0 & -\frac{1}{R}\vartheta^2 & 0 \\ \frac{1}{R}\vartheta^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.71}$$

Transferring these back to the coordinate basis we can easily see that the only non-vanishing Christoffel symbols are $\Gamma^R{}_{\Theta\Theta} = -R$ and $\Gamma^\Theta{}_{R\Theta} = \Gamma^\Theta{}_{\Theta R} = 1/R$, as expected.

Example 2.3. Let (R, Θ, Φ), $R \geqq 0, 0 \leqq \Theta \leqq \pi, 0 \leqq \Phi < 2\pi$, denote the spherical coordinates in Euclidean space. The metric reads

$$\boldsymbol{G} = \mathrm{d}R \otimes \mathrm{d}R + R^2 \mathrm{d}\Theta \otimes \mathrm{d}\Theta + R^2 \sin^2\Theta \, \mathrm{d}\Phi \otimes \mathrm{d}\Phi. \tag{2.72}$$

This leads to the choice of the following orthonormal coframes

$$\vartheta^1 = \mathrm{d}R, \quad \vartheta^2 = R\mathrm{d}\Theta, \quad \vartheta^3 = R\sin\Theta \mathrm{d}\Phi. \tag{2.73}$$

Note that

$$\mathrm{d}\vartheta^1 = 0, \quad \mathrm{d}\vartheta^2 = \frac{1}{R}\vartheta^1 \wedge \vartheta^2, \quad \mathrm{d}\vartheta^3 = -\frac{1}{R}\vartheta^3 \wedge \vartheta^1 + \frac{\cot\Theta}{R}\vartheta^2 \wedge \vartheta^3. \tag{2.74}$$

Assuming metric compatibility and using Cartan's first structural equations we find the matrix of connection 1-forms as

$$\boldsymbol{\omega} = [\omega^\alpha{}_\beta] = \begin{pmatrix} 0 & -\frac{1}{R}\vartheta^2 & -\frac{1}{R}\vartheta^3 \\ \frac{1}{R}\vartheta^2 & 0 & -\frac{\cot\Theta}{R}\vartheta^3 \\ \frac{1}{R}\vartheta^3 & \frac{\cot\Theta}{R}\vartheta^3 & 0 \end{pmatrix}. \tag{2.75}$$

Transferring these back to the coordinate basis we recover the classical Christoffel symbols.

We now show that a given coframe field with a prescribed torsion field and metric compatible determines a unique connection. STERNBERG [68] shows this for the Levi-Civita connection, but the proof can easily be extended for non-symmetric connections. Let $\vartheta^1, \ldots, \vartheta^p$ be p linearly independent 1-forms in $\mathcal{B}$ ($p \leqq n$). Now suppose that the 1-forms $\xi_1, \ldots, \xi_p$ satisfy

$$\xi_\alpha \wedge \vartheta^\alpha = 0. \tag{2.76}$$

Then according to *Cartan's Lemma* [68]

$$\xi_\alpha = \xi_{\alpha\beta}\vartheta^\beta, \quad \xi_{\alpha\beta} = \xi_{\beta\alpha}. \tag{2.77}$$

Given a torsion field, we know that a metric compatible connection satisfies $\mathcal{T}^\alpha = \mathrm{d}\vartheta^\alpha + \omega^\alpha{}_\beta \wedge \vartheta^\beta$ and $\omega^\alpha{}_\beta = -\omega^\beta{}_\alpha$. Assuming that another connection $\tilde{\omega}^\alpha{}_\beta$ satisfies these equations and denoting $\sigma^\alpha{}_\beta = \tilde{\omega}^\alpha{}_\beta - \omega^\alpha{}_\beta$, we see that

$$\sigma^\alpha{}_\beta \wedge \vartheta^\beta = 0. \tag{2.78}$$

Therefore, Cartan's Lemma tells us that $\sigma^\alpha{}_\beta = \sigma^\alpha{}_{\gamma\beta} \wedge \vartheta^\gamma$, $\sigma^\alpha{}_{\gamma\beta} = \sigma^\alpha{}_{\beta\gamma}$. But because both connections are metric compatible $\sigma^\alpha{}_\beta = -\sigma^\beta{}_\alpha$ or $\sigma^\alpha{}_{\gamma\beta} = -\sigma^\beta{}_{\gamma\alpha}$. Thus

$$\sigma^\alpha{}_{\gamma\beta} = \sigma^\alpha{}_{\beta\gamma} = -\sigma^\gamma{}_{\beta\alpha} = -\sigma^\gamma{}_{\alpha\beta} = \sigma^\beta{}_{\alpha\gamma} = \sigma^\beta{}_{\gamma\alpha} = -\sigma^\alpha{}_{\gamma\beta}. \tag{2.79}$$

Therefore, $\sigma^\alpha{}_{\gamma\beta} = 0$ and hence $\sigma^\alpha{}_\beta = 0$, that is, the connection is unique. In particular, if $\mathcal{T}^\alpha = 0$, then this unique connection is the Levi-Civita connection.

Remark 2.4. In dimension two, given a coframe field $\{\vartheta^1, \vartheta^2\}$ we see that the only nonzero connection one-form is $\omega^1{}_2$. Consequently the only nonzero curvature 2-form is $\mathcal{R}^1{}_2$. Torsion 2-forms are

$$\mathcal{T}^1 = \mathrm{d}\vartheta^1 + \omega^1{}_2 \wedge \vartheta^2, \quad \mathcal{T}^2 = \mathrm{d}\vartheta^2 - \omega^1{}_2 \wedge \vartheta^1. \tag{2.80}$$

2.3. Metrizability of an Affine Connection

Given a manifold with an affine connection, one may ask whether it is metrizable, that is, if a metric $\mathbf{G}$ exists such that $\nabla\mathbf{G} = \mathbf{0}$. In other words, the connection of the manifold respects the inner product of vectors. Physically, the question is whether a manifold can have a natural metric. If such a metric exists, then one has a natural way of measuring distances in the manifold. This is important for an elastic body as the local changes of distances determine the distribution of stresses. In other words, having the material metric one can transform the anelasticity problem to an equivalent elasticity problem. The affine connection ∇ is metrizable if there exists a full-rank symmetric second-order covariant tensor field $\mathbf{G}$ such that

$$\frac{\partial G_{IJ}}{\partial X^K} - \Gamma^M{}_{KI} G_{MJ} - \Gamma^M{}_{KJ} G_{MI} = 0. \tag{2.81}$$

This problem has been studied for both sysmmetric and non-symmetric connections by many authors (see [16,20,57,72,73] and references therein). EISENHART and Veblen [20] proved that an affine manifold $(\mathcal{B}, \nabla)$ with Riemann curvature tensor $\boldsymbol{\mathcal{R}}$ is metrizable if and only if[2]

$$\mathcal{R}^M{}_{KLJ} G_{MI} + \mathcal{R}^M{}_{KLI} G_{MJ} = 0, \tag{2.82}$$

has a non-trivial solution for $\mathbf{G}$ and any solution satisfies the following

$$\mathcal{R}^M{}_{KLJ|P} G_{MI} + \mathcal{R}^M{}_{KLI|P} G_{MJ} = 0 \quad I, J, K, L, M, P \in \{1, \ldots, n\}, \tag{2.83}$$

where $n = \dim \mathcal{B}$. The following steps lead to the most general solution for $\mathbf{G}$ [72]. Suppose $\{\mathbf{G}^{(1)}, \ldots, \mathbf{G}^{(m)}\}$ is a basis of the solution space of the linear system (2.82) and so any solution $\mathbf{G}$ has the representation

$$\mathbf{G} = \sum_{i=1}^{m} f^{(i)} \mathbf{G}^{(i)}, \tag{2.84}$$

for some functions $f^{(i)}$ defined on $\mathcal{B}$. By taking the covariant derivative of $\mathcal{R}^M{}_{KLJ} G^{(i)}_{MI} + \mathcal{R}^M{}_{KLI} G^{(i)}_{MJ} = 0$ and noting that (2.83) holds for each $\mathbf{G}^{(i)}$ one obtains

$$\mathcal{R}^M{}_{KLJ} G^{(i)}_{MI|P} + \mathcal{R}^M{}_{KLI} G^{(i)}_{MJ|P} = 0 \quad i = 1, \ldots, m. \tag{2.85}$$

This means that

$$G^{(i)}_{IJ|K} = \sum_{j=1}^{m} \psi^{(ij)}_K G^{(j)}_{IJ}, \tag{2.86}$$

for some scalar functions $\psi^{(ij)}_K$. Therefore, $\nabla \mathbf{G} = \mathbf{0}$ implies

$$\frac{\partial f^{(i)}}{\partial X^K} + \sum_{j=1}^{m} f^{(j)} \psi^{(ji)}_K = 0. \tag{2.87}$$

From the Ricci identity (2.13) and (2.83) we know that

$$G^{(i)}_{IJ|KL} - G^{(i)}_{IJ|LK} = \mathcal{R}^M{}_{KLI} G^{(i)}_{MJ} + \mathcal{R}^M{}_{KLJ} G^{(i)}_{MI} = 0. \tag{2.88}$$

Therefore, in terms of the functions $\psi^{(ij)}_K$ the condition for metrizability is

$$\frac{\partial \psi^{(ij)}_K}{\partial X^L} - \frac{\partial \psi^{(ij)}_L}{\partial X^K} + \sum_{k=1}^{m} \left(\psi^{(ik)}_K \psi^{(kj)}_L - \psi^{(ik)}_L \psi^{(kj)}_K \right) = 0 \quad i, j = 1, \ldots, m, \tag{2.89}$$

[2] Eisenhart and Veblen [20] start by looking at $G_{IJ|KL} - G_{IJ|LK}$ and assume a symmetric connection. In the case of a non-symmetric connection from (2.13) we have

$$G_{IJ|KL} - G_{IJ|LK} = \mathcal{R}^M{}_{KLJ} G_{MI} + \mathcal{R}^M{}_{KLI} G_{MJ} + T^M{}_{KL} G_{IJ|M}.$$

But because $G_{IJ|M} = 0$ the last term is identically zero, hence Eisenhart and Veblen's proof remains valid even for a non-symmetric connection.

which is the integrability condition of (2.87). Therefore, the problem reduces to finding solutions of a system of m^2 PDEs.

A flat connection according to Eisenhart and Veblen's theorem is always metrizable. Since the set of equations (2.82) is empty, the solution space is the span of independent second-order covariant symmetric tensors.

Example 2.5. Let us first start with the connection of isotropic thermoelasticity in two dimensions [60]

$$\Gamma^I{}_{JK} = \Theta^{-1}(T)\Theta'(T)\delta^I_K \frac{\partial T}{\partial X^J}, \tag{2.90}$$

where the thermal deformation gradient reads $\mathbf{F}_T = \Theta(T)\mathbf{I}$. It can be shown that the following three metrics span the solution space of (2.82):

$$\begin{aligned} \mathbf{G}^{(1)} &= \mathrm{d}X^1 \otimes \mathrm{d}X^1, \quad \mathbf{G}^{(2)} = \mathrm{d}X^2 \otimes \mathrm{d}X^2, \\ \mathbf{G}^{(3)} &= \mathrm{d}X^1 \otimes \mathrm{d}X^2 + \mathrm{d}X^2 \otimes \mathrm{d}X^1. \end{aligned} \tag{2.91}$$

Thus, the general solution is

$$\mathbf{G} = f^{(1)}\mathbf{G}^{(1)} + f^{(2)}\mathbf{G}^{(2)} + f^{(3)}\mathbf{G}^{(3)}. \tag{2.92}$$

It can be shown that

$$G^{(i)}_{IJ|K} = -2\alpha T_{,K} G^{(i)}_{IJ}, \quad i = 1, 2, 3, \tag{2.93}$$

where $\alpha(T) = \Theta^{-1}(T)\Theta'(T)$ is the coefficient of thermal expansion. Hence

$$\psi^{(ii)}_K = -2\alpha T_{,K} \quad \text{no summation on } i. \tag{2.94}$$

Therefore

$$\frac{\partial f^{(i)}}{\partial X^K} - 2\alpha T_{,K} f^{(i)} = 0 \quad i = 1, 2, 3. \tag{2.95}$$

Defining $g^{(i)} = \mathrm{e}^{-2\omega} f^{(i)}$, where $\omega' = \alpha$, the above differential equations read

$$\frac{\partial g^{(i)}}{\partial X^K} = 0 \quad \text{or} \quad g^{(i)} = C_i, \tag{2.96}$$

where C_i are constants. Therefore, the metric has the following form:

$$\begin{aligned} \mathbf{G}(\mathbf{X}, T) = C_1\mathrm{e}^{2\omega(T)}\mathrm{d}X^1 \otimes \mathrm{d}X^1 + C_2\mathrm{e}^{2\omega(T)}\mathrm{d}X^2 \otimes \mathrm{d}X^2 \\ +C_3\mathrm{e}^{2\omega(T)}\left(\mathrm{d}X^1 \otimes \mathrm{d}X^2 + \mathrm{d}X^2 \otimes \mathrm{d}X^1\right). \end{aligned} \tag{2.97}$$

If at $T = 0$ the body is a flat sheet, then $C_1 = C_2 = 1$, $C_3 = 0$, and hence

$$\mathbf{G}(\mathbf{X}, T) = \mathrm{e}^{2\omega(T)}\left(\mathrm{d}X^1 \otimes \mathrm{d}X^1 + \mathrm{d}X^2 \otimes \mathrm{d}X^2\right). \tag{2.98}$$

In dimension three, (2.93) still holds and hence the metric has the following form

$$\mathbf{G}(\mathbf{X}, T) = \mathrm{e}^{2\omega(T)} \begin{pmatrix} C_1 & C_4 & C_5 \\ C_4 & C_2 & C_6 \\ C_5 & C_6 & C_3 \end{pmatrix}. \tag{2.99}$$

for constants C_i. Again, if at $T = 0$ the body is stress free in the flat Euclidean space, we have $C_1 = C_2 = C_3 = 1$, $C_4 = C_5 = C_6 = 0$, and hence $G_{IJ}(\mathbf{X}, T) = \mathrm{e}^{2\omega(T)}\delta_{IJ}$.

3. Classical Dislocation Mechanics

Before presenting a geometric dislocation mechanics, let us briefly and critically review the classical linearized and nonlinear dislocation mechanics. This will help us fix ideas and notation and will also help us see the parallel between the classical and geometric theories more clearly.

Linearized dislocation mechanics. We start with classical linearized dislocation mechanics. We consider a domain Ω. Let us denote the tensor of elastic distortions by $\boldsymbol{\beta}_e$ [39,41,65]. Given two nearby points $d\mathbf{x}$ apart in Ω, the change in the displacements $\delta\mathbf{u}$ is written as $\delta\mathbf{u} = \boldsymbol{\beta}_e d\mathbf{x}$. The strain tensor can be written as $\boldsymbol{\varepsilon} = \frac{1}{2}\left(\boldsymbol{\beta}_e + \boldsymbol{\beta}_e^{\mathsf{T}}\right) = \boldsymbol{\beta}_e^{\mathsf{S}}$. The tensor of incompatibility is defined as

$$\boldsymbol{\eta} = \mathsf{Inc}(\boldsymbol{\varepsilon}) := \nabla \times \nabla \times \boldsymbol{\varepsilon} = \mathsf{Curl} \circ \mathsf{Curl}\boldsymbol{\varepsilon}. \tag{3.1}$$

The Burgers vector associated with a closed curve $\mathcal{C} = \partial\Omega$ is defined as[3]

$$\mathbf{b} = -\int_{\mathcal{C}} \boldsymbol{\beta}_e d\mathbf{x} = -\int_{\Omega} \mathsf{Curl}\boldsymbol{\beta}_e \cdot \mathbf{n} da. \tag{3.2}$$

We assume that the domain of interest is simply-connected, that is, its homotopy group and consequently its first homology group are trivial. This means that a given closed curve is the boundary of some 2-submanifold. We now define the dislocation density tensor as [59]

$$\boldsymbol{\alpha} = -\mathsf{Curl}\boldsymbol{\beta}_e. \tag{3.3}$$

This immediately implies that

$$\mathsf{Div}\boldsymbol{\alpha} = \mathbf{0}. \tag{3.4}$$

Now the incompatibility tensor in terms of the dislocation density tensor is written as

$$\boldsymbol{\eta} = -\mathsf{Curl}\left[\frac{\boldsymbol{\alpha} + \boldsymbol{\alpha}^{\mathsf{T}}}{2}\right] = -\left(\mathsf{Curl}\boldsymbol{\alpha}\right)^{\mathsf{S}}. \tag{3.5}$$

In a simply-connected domain a stress-free dislocation density distribution corresponds to $\boldsymbol{\eta} = \mathbf{0}$. In the linearized setting the total distortion can be additively decomposed into elastic and plastic parts, that is, $\boldsymbol{\beta} = \boldsymbol{\beta}_e + \boldsymbol{\beta}_p$. The total distortion being compatible implies that

$$\boldsymbol{\alpha} = -\mathsf{Curl}\boldsymbol{\beta}_e = \mathsf{Curl}\boldsymbol{\beta}_p. \tag{3.6}$$

See [28] for some concrete examples of zero-stress dislocations distributions in the linearized setting.

[3] In components, $(\mathsf{Curl}\boldsymbol{\varepsilon})_{AB} = \varepsilon_{AMN}\varepsilon_{BN,M}$.

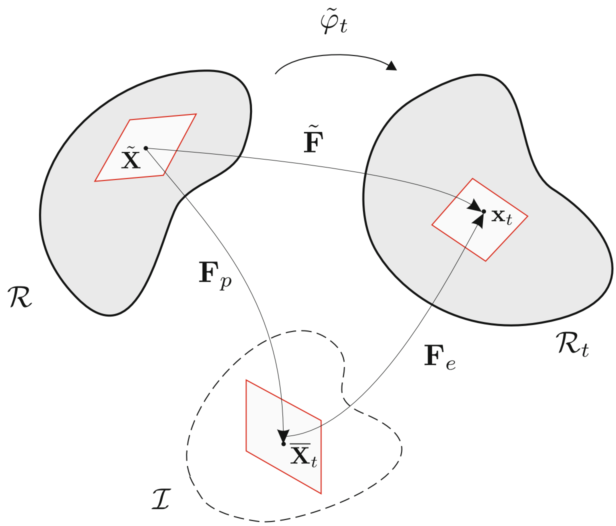

Fig. 2. Intermediate configuration in phenomenological plasticity. Note that the reference configuration $\mathcal{R}$ is residually stressed, in general, and $\mathcal{R}_t$ is the deformed (current) configuration

Nonlinear dislocation mechanics. The starting point in any phenomenological theory of nonlinear plasticity is to assume a decomposition $\mathbf{F} = \mathbf{F}_e\mathbf{F}_p$, which was originally proposed in [8,39], and [47]. Using this decomposition one then introduces an "intermediate" configuration as shown in Fig. 2. Usually, it is assumed that the reference and the final deformed bodies are embedded in a Euclidean space. Intermediate configuration is not compatible and is understood as an auxiliary configuration defined locally. In classical nonlinear dislocation mechanics, given an oriented surface Ω in the reference configuration the total Burgers vector of all the dislocations crossing Ω is calculated using the dislocation density tensor $\boldsymbol{\alpha}$ as

$$b^A(\Omega) = \int_\Omega \alpha^{AB} N_B \mathrm{d}A, \tag{3.7}$$

where $\mathbf{N}^\sharp$ is the unit normal vector to Ω.[4] Let us next critically reexamine this definition.

Given a closed curve γ in the current configuration, the Burgers vector is defined as [46][5]

[4] Note that in an orthogonal coordinate basis with one of the basis vectors parallel to the Burgers vector, for a screw dislocation $[\alpha^{AB}]$ is a diagonal matrix while for an edge dislocation all the diagonal elements of $[\alpha^{AB}]$ are zero.

[5] Here we mention [46] as an example; similar definitions can be seen in many other works.

$$\mathbf{b} = -\int_{\gamma} \mathbf{F}_e^{-1} \mathrm{d}\mathbf{x}. \tag{3.8}$$

Or in components

$$b^{\alpha} = -\int_{\gamma} \left(\mathbf{F}_e^{-1}\right)^{\alpha}{}_{b}\, \mathrm{d}x^{b}. \tag{3.9}$$

Here, it is assumed that $\{x^a\}$ and $\{U^{\alpha}\}$ are local charts for current and "intermediate" configurations, respectively. Then Le and Stumpf mention that if this integral vanishes for all closed curves, the elastic deformation is compatible. One should note that b^{α} explicitly depends on γ. Parametrizing the curve by $s \in [0, \ell]$, we can rewrite (3.9) as

$$b^{\alpha} = -\int_{0}^{\ell} \left[\mathbf{F}_e^{-1}(\mathbf{x}(s))\right]^{\alpha}{}_{b}\, \dot{\mathbf{x}}^{b}(s)\mathrm{d}s. \tag{3.10}$$

Let us denote the preimage of $\mathbf{x}(s)$ in the "intermediate" configuration by $\overline{\mathbf{X}}(s)$ and the "intermediate" space by $\mathcal{I}$. Note that for each s, $\mathbf{F}_e^{-1}(x(s))\dot{\mathbf{x}}(s) \in T_{\overline{\mathbf{X}}(s)}\mathcal{I}$. This means that (3.9) makes sense only if the "intermediate" configuration is a linear space, which is not the case.[6] Assuming that (3.9) is the "Burgers vector", Le and Stumpf [46] show that (using Stokes' theorem)

$$b^{\alpha} = \int_{\mathcal{A}} \alpha^{\alpha}{}_{bc}\, (\mathrm{d}x^{b} \wedge \mathrm{d}x^{c}), \tag{3.11}$$

where

$$\alpha^{\alpha}{}_{bc} = \frac{\partial \left(F_e^{-1}\right)_{b}{}^{\alpha}}{\partial x^{c}} - \frac{\partial \left(F_e^{-1}\right)_{c}{}^{\alpha}}{\partial x^{b}}, \tag{3.12}$$

is the dislocation density tensor.[7] Then they incorrectly conclude that $b^{\alpha} = \frac{1}{2}\alpha^{\alpha}{}_{bc}\, \mathrm{d}x^{b} \wedge \mathrm{d}x^{c} = \alpha^{\alpha}{}_{bc}\, (\mathrm{d}x^{b} \wedge \mathrm{d}x^{c})$, ignoring the area of their infinitesimal circuit γ.

Remark 3.1. Ozakin and Yavari [61] showed that at a point $\mathbf{X}$ in the material manifold $\mathcal{B}$, torsion 2-form acting on a 2-plane section of $T_{\mathbf{X}}\mathcal{B}$ gives the density of the Burgers vector.

Remark 3.2. Acharya and Bassani [2] realized the importance of the area of the enclosing surface in (3.9) and called the resulting vector "cumulative Burgers vector". In classical three-dimensional continuum mechanics, all second-order tensors are meant to be linear transformations on V_3—the translation space of E_3; the latter being the ambient three-dimensional Euclidean space. V_3 is a three-dimensional vector space endowed with the standard Euclidean inner product. Thus, $\mathbf{F}_e^{-1}(\mathbf{x}) : V_3 \to V_3$, $\mathbf{x} \in \varphi(\mathcal{B})$. With this understanding

[6] Integrating a vector field is not intrinsically meaningful as parallel transport is path dependent in the presence of curvature. When a manifold is flat a vector field can be integrated but the integration explicitly depends on the connection used in defining parallel transport.

[7] One can equivalently write the dislocation density tensor in terms of $\mathbf{F}_p$.

$$\mathbf{b} = -\int_{\gamma} \mathbf{F}_e^{-1} d\mathbf{x}, \tag{3.13}$$

makes sense as a line integral in V_3. For example, invoke a fixed rectangular Cartesian coordinate system in E_3 with respect to its natural (coordinate) basis, represent $\mathbf{F}_e^{-1}$ as a matrix field on $\varphi(\mathcal{B})$ and γ as a curve in $\mathbb{R}^3$. The above procedure works because one thinks that the range of $\mathbf{F}_e^{-1}(\mathbf{x})$ for each $\mathbf{x} \in \varphi(\mathcal{B})$ is the same vector space V_3 independent of $\mathbf{x}$. For instance, when $\mathbf{F}_e^{-1}$ is compatible on $\varphi(\mathcal{B})$, let φ_e be the deformation of $\varphi(\mathcal{B})$ with $T\varphi_e = \mathbf{F}_e^{-1}$. Geometrically, $\mathbf{F}_e^{-1} : T_{\mathbf{x}}\varphi(\mathcal{B}) \to T_{\tilde{\mathbf{X}}}(\varphi_e \circ \varphi)(\mathcal{B})$, where $\tilde{\mathbf{X}} = \varphi_e(\mathbf{x})$, and in this three-dimensional setting we know that physically $T_{\mathbf{x}}\varphi(\mathcal{B}) = T_{\tilde{\mathbf{X}}}(\varphi_e \circ \varphi)(\mathcal{B}) = V_3$ (they contain exactly the same set of vectors) and this relationship is independent of $\mathbf{x}$. Thus, in this case of three-dimensional continuum mechanics, the above procedure is physically meaningful and mathematically unambiguous. In the geometric approach, this freedom of identifying $T_{\mathbf{x}}\varphi(\mathcal{B})$ and $T_{\tilde{\mathbf{X}}}(\varphi_e \circ \varphi)(\mathcal{B})$ and the independence of this identification of $\mathbf{x} \in \varphi(\mathcal{B})$ cannot be exercised. Then, the formalism of Ozakin and Yavari [61] needs to be invoked to define the density of the Burgers vector. We note here that such a formalism would remain valid for defining the Burgers vector for a curve on a lower dimensional body like a shell or a membrane whereas the three-dimensional formalism would not without modification (as can be seen even in the compatible case).

Note that a defect distribution, in general, leads to residual stresses essentially because the body is constrained to deform in Euclidean space. If one partitions the body into small pieces, each piece will individually relax, but it is impossible to realize a relaxed state for the whole body by combining these pieces in Euclidean space. Any attempt to reconstruct the body by sticking the particles together will induce deformations on them, and will result in stresses. An imaginary relaxed configuration for the body is incompatible with the geometry of Euclidean space. Consider one of these small relaxed pieces. The process of relaxation after the piece is cut corresponds to a linear deformation of this piece (linear, since the piece is small). Let us call this deformation $\mathbf{F}_p$. If this piece is deformed in some arbitrary way after the relaxation, one can calculate the induced stresses by using the tangent map of this deformation mapping in the constitutive relation. In order to calculate the stresses induced for a given deformation of the body, we focus our attention on a small piece. The deformation gradient of the body at this piece $\mathbf{F}$ can be decomposed as $\mathbf{F} = \mathbf{F}_e\mathbf{F}_p$, where, by definition, $\mathbf{F}_e = \mathbf{F}\mathbf{F}_p^{-1}$. Thus, as far as this piece is concerned, the deformation of the body consists of a relaxation, followed by a linear deformation given by $\mathbf{F}_e$. The stresses induced on this piece, for an arbitrary deformation of the body, can be calculated by substituting $\mathbf{F}_e$ in the constitutive relation. Note that $\mathbf{F}_e$ and $\mathbf{F}_p$ are not necessarily true deformation gradients in the sense that one cannot necessarily find deformations φ_e and φ_p whose tangent maps are given by $\mathbf{F}_e$ and $\mathbf{F}_p$, respectively. This is due precisely to the incompatibility mentioned above. In the sequel, we will see that an "intermediate" configuration is not necessary; one can define a global stress-free reference manifold instead of working with local stress-free configurations.

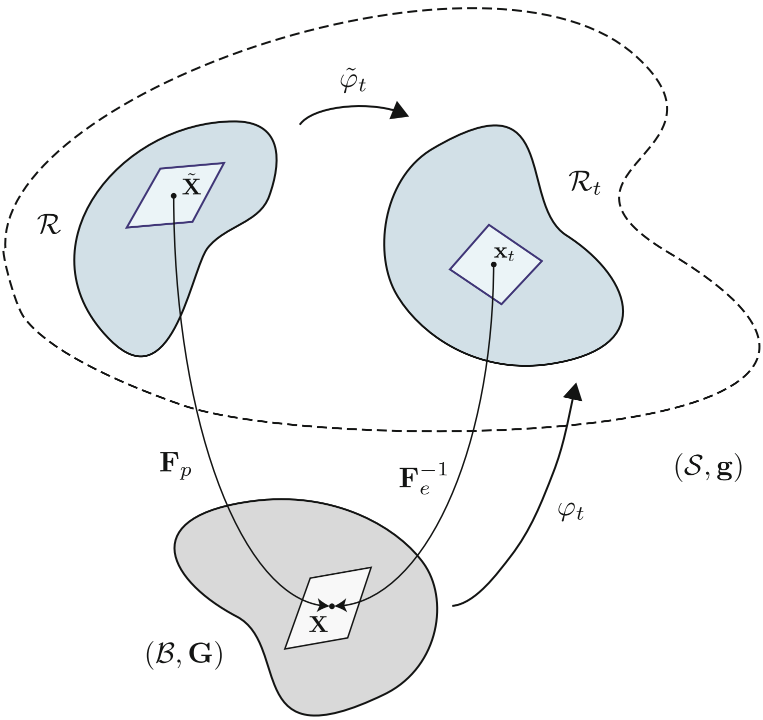

Fig. 3. Connection between the geometric theory and the classical $\mathbf{F} = \mathbf{F}_e\mathbf{F}_p$ decomposition. Push-forward of $T_{\tilde{\mathbf{X}}}\mathcal{R}$ and pull-back of $T_{\mathbf{x}}\mathcal{R}_t$ are identified with $T_{\mathbf{X}}\mathcal{B}$. Note that the reference configuration $\mathcal{R}$ and the current configuration $\mathcal{R}_t$ are embeddings of the material manifold $\mathcal{B}$ into the ambient Riemannian manifold $\mathcal{S}$

4. Dislocation Mechanics and Cartan's Moving Frames

In this section we show that $\mathbf{F}_p$ naturally defines a moving frame for the material manifold.[8] Let us assume $\mathbf{F} = \mathbf{F}_e\mathbf{F}_p$ is given. $\mathbf{F}_e^{-1}$ maps a stressed tangent space (or a local configuration) to a relaxed tangent space (or local configuration). Equivalently, $\mathbf{F}_p$ maps a stressed reference tangent space (or a local reference configuration) to a relaxed reference tangent space (or local relaxed reference configuration). Now $\mathbf{F}_p$ acting on a local basis in $\mathcal{R}$ gives a local basis in the relaxed local configuration [2]. We assume that this is a basis for the tangent space of the material manifold. In other words, we identify the relaxed tangent space with the tangent space of the material manifold (see Fig. 3). This will be explained in more detail in the sequel.

The dislocated body is stress free in the material manifold by construction. Let us consider a coordinate basis $\{X^A\}$[9] for the material manifold $\mathcal{B}$ and a basis $\{\mathbf{E}_{\bar{A}}\}$

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[8] We are grateful to Amit Acharya for his comments regarding the connection between our geometric theory and the $\mathbf{F} = \mathbf{F}_e\mathbf{F}_p$ decomposition.

[9] A coordinate chart $\{X^A\}$ induces a coordinate basis $\{\partial_A = \frac{\partial}{\partial X^A}\}$ for the tangent space [1]. A moving frame is a basis for the tangent space but does not necessarily come from a coordinate chart, that is, it may be a non-coordinate basis.

(with dual basis $\{\mathbf{E}^{\bar{A}}\}$) for $T_{\tilde{\mathbf{X}}}\mathcal{R}$. Note that $\mathbf{F}_p : T_{\tilde{\mathbf{X}}}\mathcal{R} \to T_{\mathbf{X}}\mathcal{B}$ and hence it has the following two representations with respect to $\{\partial_A\}$ and $\mathbf{e}_\alpha$, respectively

$$\mathbf{F}_p = \left(F_p\right)^A{}_{\bar{A}}\, \partial_A \otimes \mathbf{E}^{\bar{A}} = \left(F_p\right)^\alpha{}_{\bar{A}}\, \mathbf{e}_\alpha \otimes \mathbf{E}^{\bar{A}}. \tag{4.1}$$

We assume that the basis $\mathbf{E}_{\bar{A}}$ is such that $\mathbf{F}_p \cdot \mathbf{E}_1 = \partial_1$, etc., that is $\mathbf{F}_p = \delta^A_{\bar{A}}\, \partial_A \otimes \mathbf{E}^{\bar{A}}$. Hence, given $\mathbf{F}_p$, it defines the following frame and coframe fields

$$\mathbf{e}_\alpha = \left(F_p^{-1}\right)_\alpha{}^A \partial_A, \quad \vartheta^\alpha = (F_p)^\alpha{}_A \mathrm{d}X^A. \tag{4.2}$$

Material metric in the coordinate basis has the following components:

$$G_{AB} = \left(F_p\right)^\alpha{}_A \left(F_p\right)^\beta{}_B \delta_{\alpha\beta}. \tag{4.3}$$

We demand absolute parallelism in $(\mathcal{B}, \nabla, \mathbf{G})$, that is, we equip the material manifold with an evolving connection (compatible with the metric) such that the frame field is everywhere parallel. This connection is the Weitzenböck connection with the following components in the coordinate basis

$$\Gamma^A{}_{BC} = \left(F_p^{-1}\right)_\alpha{}^A \partial_B \left(F_p\right)^\alpha{}_C. \tag{4.4}$$

Using Cartan's first structural equations, torsion 2-form is

$$\boldsymbol{\mathcal{T}} = \mathbf{e}_\alpha \otimes \mathcal{T}^\alpha = \mathbf{e}_\alpha \otimes \left(\mathrm{d}\vartheta^\alpha + \omega^\alpha{}_\beta \wedge \vartheta^\beta\right) = \mathbf{e}_\alpha \otimes \mathrm{d}\vartheta^\alpha. \tag{4.5}$$

This can be written in the coordinate basis as

$$\begin{aligned} \boldsymbol{\mathcal{T}} &= \partial_A \otimes \left(F_p^{-1}\right)_\alpha{}^A d\left[(F_p)^\alpha{}_C \mathrm{d}X^C\right] \\ &= \partial_A \otimes \left(F_p^{-1}\right)_\alpha{}^A \partial_B (F_p)^\alpha{}_C\, \mathrm{d}X^B \wedge \mathrm{d}X^C \\ &= \partial_A \otimes \left(F_p^{-1}\right)_\alpha{}^A \left[\partial_B (F_p)^\alpha{}_C - \partial_C (F_p)^\alpha{}_B\right] \left(\mathrm{d}X^B \wedge \mathrm{d}X^C\right), \end{aligned} \tag{4.6}$$

where $\{\left(\mathrm{d}X^B \wedge \mathrm{d}X^C\right)\} = \{\mathrm{d}X^B \wedge \mathrm{d}X^C\}_{B<C}$ is a basis for 2-forms, that is, $Q_{BC}\left(\mathrm{d}X^B \wedge \mathrm{d}X^C\right) = \sum_{B<C} Q_{BC} \mathrm{d}X^B \wedge \mathrm{d}X^C$. For a dislocated body the material connection is flat, hence the first Bianchi identity reads (Weitzenböck connection is flat by definition)

$$D\mathcal{T}^\alpha = \mathrm{d}\mathcal{T}^\alpha + \omega^\alpha{}_\beta \wedge \vartheta^\beta = \mathrm{d}\mathrm{d}\vartheta^\alpha = 0. \tag{4.7}$$

Note that given a torsion 2-form $\boldsymbol{\mathcal{T}}$ the corresponding dislocation density tensor is defined as

$$\boldsymbol{\alpha} = (*\boldsymbol{\mathcal{T}})^\sharp. \tag{4.8}$$

We know that $D\boldsymbol{\mathcal{T}} = (\mathrm{Div}\,\boldsymbol{\alpha}) \otimes \mu$ and hence the first Bianchi identity is equivalent to

$$\mathrm{Div}\,\boldsymbol{\alpha} = \mathbf{0}. \tag{4.9}$$

The explicit relation between torsion 2-form and dislocation density tensor. Note that $\boldsymbol{\mathcal{T}}$ is a vector-valued 2-form and hence $*_2\boldsymbol{\mathcal{T}}$ is vector-valued 1-form. In components

$$*_2\,\boldsymbol{\mathcal{T}} = \partial_A \otimes (*\mathcal{T}^A)^B = \partial_A \otimes \left(\frac{1}{2} T^A{}_{CD}\varepsilon^{CD}{}_B\right) \mathrm{d}X^B, \tag{4.10}$$

where ε_{ABC} is the Levi-Civita tensor. Therefore, $(*_2\boldsymbol{\mathcal{T}})^{\sharp_2}$ in components reads

$$(*_2\boldsymbol{\mathcal{T}})^{\sharp_2} = \left(\frac{1}{2} T^A{}_{CD}\varepsilon^{BCD}\right) \partial_A \otimes \partial_B. \tag{4.11}$$

This is the dislocation density tensor $\boldsymbol{\alpha}$, which is a $\binom{2}{0}$-tensor with components $\alpha^{AB} = \frac{1}{2} T^A{}_{CD}\varepsilon^{BCD}$. Equivalently, $T^A{}_{BC} = \alpha^{AM}\varepsilon_{MBC}$.

Parallelizable Manifolds, Dislocation Mechanics, and Relation with $\mathbf{F} = \mathbf{F}_e\mathbf{F}_p$. Here we show that in multiplicative plasticity one can combine the reference and "intermediate" configurations into a parallelizable material manifold. Let us start with a coordinate basis $\partial_I = \frac{\partial}{\partial X^I}$ and its dual $\{\mathrm{d}X^I\}$. Define a moving coframe by $\vartheta^\alpha = \left(F_p\right)^\alpha{}_I \mathrm{d}X^I$. This means that the moving frame is defined as $\mathbf{e}_\alpha = \left(F_p^{-1}\right)_\alpha{}^I \partial_I$. Assuming that connection 1-forms $\omega^\beta{}_\alpha$ are given we have $\nabla\mathbf{e}_\alpha = \omega^\beta{}_\alpha \mathbf{e}_\beta$. Torsion 2-form is defined as

$$\mathcal{T}^\alpha = \mathrm{d}\vartheta^\alpha + \omega^\alpha{}_\beta \wedge \vartheta^\beta = \left[\frac{\partial \left(F_p\right)^\alpha{}_I}{\partial X^J} + \omega^\alpha{}_{\beta J}\left(F_p\right)^\beta{}_I\right] \mathrm{d}X^J \wedge \mathrm{d}X^I. \tag{4.12}$$

We know that

$$\omega^\beta{}_{\alpha K} = \left(F_p\right)^\beta{}_J \left(F_p^{-1}\right)_\alpha{}^I\, \Gamma^J{}_{IK} + \left(F_p\right)^\beta{}_I \frac{\partial \left(F_p^{-1}\right)_\alpha{}^I}{\partial X^K}. \tag{4.13}$$

Requiring that the frame $\mathbf{e}_\alpha$ be everywhere parallel is equivalent to

$$\left(F_p^{-1}\right)_\alpha{}^I{}_{|J} = 0. \tag{4.14}$$

This gives

$$\Gamma^I{}_{JK} = \left(F_p^{-1}\right)_\alpha{}^I \frac{\partial \left(F_p\right)^\alpha{}_K}{\partial X^J}. \tag{4.15}$$

Note that $\mathcal{T}^\alpha = \left(F_p\right)^\alpha{}_I \mathcal{T}^I$, where

$$\mathcal{T}^I = T^I{}_{JK}(\mathrm{d}X^J \wedge \mathrm{d}X^K). \tag{4.16}$$

Thus

$$\mathcal{T}^\alpha = \left[\frac{\partial \left(F_p\right)^\alpha{}_K}{\partial X^J} - \frac{\partial \left(F_p\right)^\alpha{}_J}{\partial X^K}\right] (\mathrm{d}X^J \wedge \mathrm{d}X^K). \tag{4.17}$$

And

$$
\begin{aligned}
\mathcal{T}^I &= \left(F_p^{-1}\right)_\alpha{}^I \, \mathcal{T}^\alpha \\
&= \left(F_p^{-1}\right)_\alpha{}^I \left[\frac{\partial \left(F_p\right)^\alpha{}_K}{\partial X^J} - \frac{\partial \left(F_p\right)^\alpha{}_J}{\partial X^K} \right] (\mathrm{d}X^J \wedge \mathrm{d}X^K). \qquad (4.18)
\end{aligned}
$$

In summary, instead of working with a Euclidean reference manifold and an "intermediate" configuration, one can assume that the material manifold is equipped with a Weitzenböck connection. The material manifold can be described by Cartan's moving frames $\{\mathbf{e}_\alpha\}$ and coframes $\{\vartheta^\alpha\}$. Using this representation of material manifold, nonlinear dislocation mechanics has a structure very similar to that of classical nonlinear elasticity; the main differences are the non-Euclidean nature of the reference configuration and its evolution in time.

4.1. Zero-Stress (Impotent) Dislocation Distributions

It may happen that a nontrivial distribution of dislocations, that is, when $\mathbf{F}_p \neq \mathbf{0}$, or non-vanishing dislocation density tensor leads to zero residual stresses. Here, we characterize these zero-stress or impotent dislocation distributions. Given a field of plastic deformation gradients $\mathbf{F}_p$, the material connection is written as

$$
\Gamma^I{}_{JK} = \left(F_p^{-1}\right)_\alpha{}^I \left(F_p\right)^\alpha{}_{J,K}. \qquad (4.19)
$$

The material metric is $G_{IJ} = \left(F_p\right)^\alpha{}_I \left(F_p\right)^\beta{}_J \, \delta_{\alpha\beta}$. Note that by construction $G_{IJ|K} = 0$.

Impotency in terms of $\mathbf{F}_p$. In the orthonormal frame $\{\mathbf{e}_\alpha\}$ the Weizentböck connection 1-forms vanish, that is, $\omega^\alpha{}_\beta = 0$. This means that

$$
\mathcal{T}^\alpha = \mathrm{d}\vartheta^\alpha + \omega^\alpha{}_\beta \wedge \vartheta^\beta = \mathrm{d}\vartheta^\alpha. \qquad (4.20)
$$

If torsion 2-form vanishes, that is, $\mathrm{d}\vartheta^\alpha = 0$ then according to Poincaré's Lemma we locally (globally if the body is simply connected) have $\vartheta^\alpha = df^\alpha$ for some 0-forms f^α. This means that the plastic distortions are compatible and hence impotent. From (2.66) vanishing torsion in a coordinate basis implies

$$
\partial_A (F_p)^\alpha{}_B = \partial_B (F_p)^\alpha{}_A. \qquad (4.21)
$$

This is the familiar $\mathsf{Curl}\ \mathbf{F}_p = \mathbf{0}$. Let us now show that vanishing torsion of the Weitzenböck connection implies flatness of the underlying Riemannian material manifold.

Lemma 4.1. *If torsion of the Weitzenböck connection vanishes, then the underlying Riemannian manifold is locally flat.*

Proof. For the Levi-Civita connection we have

$$d\vartheta^\alpha + \overline{\omega}^\alpha{}_\beta \wedge \vartheta^\beta = \overline{\omega}^\alpha{}_\beta \wedge \vartheta^\beta = 0. \tag{4.22}$$

Using Cartan's Lemma and noticing that because of metric compatibility $\overline{\omega}^\alpha{}_\beta = -\overline{\omega}^\beta{}_\alpha$ we conclude that $\overline{\omega}^\alpha{}_\beta = 0$ (very similar to the proof that was given for uniqueness of metric compatible connection for a given torsion field). Thus

$$\overline{\mathcal{R}}^\alpha{}_\beta = d\overline{\omega}^\alpha{}_\beta + \overline{\omega}^\alpha{}_\gamma \wedge \overline{\omega}^\gamma{}_\beta = 0. \tag{4.23}$$

This shows that the underlying Riemannian material manifold is (locally) flat.

□

Remark 4.2. Note that the converse of this lemma is not true, that is, there are non-vanishing torsion distributions which are zero stress. We will find several examples in the sequel.[10]

Example 4.3. We consider the two examples given in [2]:

$$\text{Case 1} : \mathbf{F}_p = \mathbf{I} + \gamma(X^2)\mathbf{E}_1 \otimes \mathbf{E}_2, \tag{4.24}$$

$$\text{Case 2} : \mathbf{F}_p = \mathbf{I} + \gamma(X^2)\mathbf{E}_2 \otimes \mathbf{E}_1. \tag{4.25}$$

For Case 1, it can be shown that the only nonzero Weitzenböck connection coefficient is $\Gamma^1{}_{22} = \gamma'(X^2)$, that is, the torsion tensor vanishes identically. We have the following material metric:

$$\mathbf{G} = \begin{pmatrix} 1 & \gamma & 0 \\ \gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.26}$$

It can also be shown that the only nonzero Levi-Civita connection coefficient is $\overline{\Gamma}^1{}_{22} = 1$. It is seen that the Riemannian curvature tensor identically vanishes, that is, $\mathbf{F}_p$ in Case 1 is impotent. For Case 2, the only nonzero Weitzenböck connection coefficient is $\Gamma^2{}_{21} = \gamma'(X^2)$, and hence the only nonzero torsion coefficients are $T^2{}_{21} = -T^2{}_{12} = \gamma'(X^2)$, that is, $\mathbf{F}_p$ in Case 2 is not impotent. We have the following material metric:

$$\mathbf{G} = \begin{pmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.27}$$

Remark 4.4. We can use Cartan's moving frames as follows. Case 1: We have the following moving coframe field

[10] We should mention that in the linearized setting the dislocation distributions for which $\boldsymbol{\eta} = \mathbf{0}$ were called impotent or stress-free dislocation distributions by Mura [54]. Note that if $\boldsymbol{\beta}_p^{\mathsf{S}} = \mathbf{0}$ then $\boldsymbol{\varepsilon} = \mathbf{0}$ and hence $\boldsymbol{\eta} = \mathbf{0}$. However, the set of zero-stress dislocation distributions is larger.

$$\vartheta^1 = \mathrm{d}X^1 + \gamma(X^2)\mathrm{d}X^2, \quad \vartheta^2 = \mathrm{d}X^2, \quad \vartheta^3 = \mathrm{d}X^3. \tag{4.28}$$

Thus, $\mathrm{d}\vartheta^1 = \mathrm{d}\vartheta^2 = \mathrm{d}\vartheta^3 = 0$. This means that $\mathcal{T}^\alpha = 0$ and Lemma 4.1 tells us that the Levi-Civita connection is flat, that is, $\mathbf{F}_p$ is impotent. Case 2: We have the following moving coframe field

$$\vartheta^1 = \mathrm{d}X^1, \quad \vartheta^2 = \gamma(X^2)\mathrm{d}X^1 + \mathrm{d}X^2, \quad \vartheta^3 = \mathrm{d}X^3. \tag{4.29}$$

This means that $\mathrm{d}\vartheta^1 = \mathrm{d}\vartheta^3 = 0$ and $\mathrm{d}\vartheta^2 = -\gamma'(X^2)\mathrm{d}X^1 \wedge \mathrm{d}X^2$. The Levi-Civita connections are obtained as

$$\overline{\omega}^1{}_2 = -\gamma'(X^2)\vartheta^2, \quad \overline{\omega}^2{}_3 = \overline{\omega}^3{}_1 = 0. \tag{4.30}$$

Therefore

$$\overline{\mathcal{R}}^1{}_2 = \mathrm{d}\overline{\omega}^1{}_2 = -(\gamma\gamma'' + \gamma'^2)\vartheta^1 \wedge \vartheta^2, \quad \overline{\mathcal{R}}^2{}_3 = \overline{\mathcal{R}}^3{}_1 = 0, \tag{4.31}$$

that is, the Riemannian material manifold is not flat, unless $\gamma\gamma'' + \gamma'^2 = 0$.

Impotency in terms of torsion tensor. Now one may ask which dislocation distributions are zero stress. In the geometric framework we are given the torsion tensor $T^A{}_{BC}$ in a coordinate basis $\{X^A\}$. Let us now look at (2.22). Given the torsion tensor, the contorsion tensor is defined as

$$K^A{}_{BC} = \frac{1}{2}\left(T^A{}_{BC} + G_{BM}G^{AN}T^M{}_{NC} + G_{CM}G^{AN}T^M{}_{NB}\right). \tag{4.32}$$

Note that the metric tensor is an unknown at this point.[11] For a distributed dislocation curvature tensor (and consequently Ricci curvature tensor) of the non-symmetric connection vanishes, hence from (2.22) we have

$$0 = \overline{R}_{AB} + K^M{}_{AB|M} - K^M{}_{MB|A} + K^N{}_{NM}K^M{}_{AB} - K^N{}_{AM}K^M{}_{NB}. \tag{4.33}$$

Note that because the Ricci curvature is symmetric $K^M{}_{AB|M} - K^M{}_{MB|A} + K^N{}_{NM}K^M{}_{AB} - K^N{}_{AM}K^M{}_{NB}$ must be symmetric as well. The torsion distribution $T^A{}_{BC}$ is zero stress if the Riemannian material manifold is flat, which for a three-dimensional manifold means $\overline{R}_{AB} = 0$. Therefore, the following characterizes the impotent torsion distributions: *a torsion distribution is impotent if the symmetric part of the following system of nonlinear PDEs has a solution for G_{AB} and its anti-symmetric part vanishes.*

$$K^M{}_{AB|M} - K^M{}_{MB|A} + K^N{}_{NM}K^M{}_{AB} - K^N{}_{AM}K^M{}_{NB} = 0. \tag{4.34}$$

In dimension two we have the same result for the scalar curvature, that is, the following nonlinear PDE should have a solution for G_{AB}.

$$G^{AB}\left(K^M{}_{AB|M} - K^M{}_{MB|A} + K^N{}_{NM}K^M{}_{AB} - K^N{}_{AM}K^M{}_{NB}\right) = 0. \tag{4.35}$$

Example 4.5. Let us consider an isotropic distribution of screw dislocations, that is, the torsion tensor is completely anti-symmetric. In this case $K^A{}_{BC} = \frac{1}{2}T^A{}_{BC}$, and $K^M{}_{MB} = 0$. Therefore, (4.34) is simplified to read

[11] Here, we are given a torsion tensor that comes from an (a priori unknown) Weitzenböck connection, which is metric compatible. However, the metric compatible with the material connection is not known a priori.

$$2T^M{}_{AB|M} - T^N{}_{AM}T^M{}_{NB} = 0. \tag{4.36}$$

It is seen that in this special case, the material metric does not enter the impotency equations. Note that the first term is anti-symmetric in (A, B) while the second one is symmetric. Therefore, each should vanish separately, that is, the impotency equations read

$$T^M{}_{AB|M} = 0, \quad T^N{}_{AM}T^M{}_{NB} = 0. \tag{4.37}$$

A completely anti-symmetric torsion tensor can be written as $T^A{}_{BC} = \mathfrak{T}\varepsilon^A{}_{BC}$, where $\mathfrak{T}$ is a scalar. The above impotency equations then read

$$\mathfrak{T}_{,M}\varepsilon^M{}_{AB} = 0, \quad \mathfrak{T}^2\varepsilon^N{}_{AM}\varepsilon^M{}_{NB} = \mathfrak{T}^2 G_{AB} = 0. \tag{4.38}$$

Therefore, $\mathfrak{T} = 0$, that is, a non-vanishing isotropic distribution of screw dislocations cannot be zero-stress.

4.2. Some Non-Trivial Zero-Stress Dislocation Distributions in Three Dimensions

Let us next present some non-trivial examples of zero-stress dislocation distributions. The idea is to start with an orthonormal coframe field for the flat three space and then try to construct a flat connection for a given torsion field. If such a flat connection exists, the corresponding torsion field (dislocation distribution) is zero-stress.

Cartesian coframe. We start with the following orthonormal moving coframe field

$$\vartheta^1 = dX, \quad \vartheta^2 = dY, \quad \vartheta^3 = dZ. \tag{4.39}$$

Note that the metric is $\mathbf{G} = dX \otimes dX + dY \otimes dY + dZ \otimes dZ$. We know that the Levi-Civita connection of this metric is flat. Now, if a given torsion distribution has a flat connection in this coframe field, then the given torsion distribution is zero-stress. Let us first start with a distribution of screw distributions (not necessarily isotropic), that is

$$\mathcal{T}^1 = \xi\vartheta^2 \wedge \vartheta^3, \quad \mathcal{T}^2 = \eta\vartheta^3 \wedge \vartheta^1, \quad \mathcal{T}^3 = \lambda\vartheta^1 \wedge \vartheta^2, \tag{4.40}$$

for some functions ξ, η, and λ of (X, Y, Z). Cartan's first structural equations give us the following connection 1-forms

$$\omega^1{}_2 = \frac{-\xi - \eta + \lambda}{2}\vartheta^3, \quad \omega^2{}_3 = \frac{\xi - \eta - \lambda}{2}\vartheta^1, \quad \omega^3{}_1 = \frac{-\xi + \eta - \lambda}{2}\vartheta^2. \tag{4.41}$$

From Cartan's second structural equations curvature 2-form vanishes if and only if $\xi = \eta = \lambda = 0$, that is, in this moving frame field any non-zero screw dislocation distribution induces stresses.

Now let us look at edge dislocations and assume that torsion forms are given as

$$\begin{aligned}&\mathcal{T}^1 = A\,\vartheta^1 \wedge \vartheta^2 + B\,\vartheta^3 \wedge \vartheta^1, \quad \mathcal{T}^2 = C\,\vartheta^1 \wedge \vartheta^2 + D\,\vartheta^2 \wedge \vartheta^3,\\ &\mathcal{T}^3 = E\,\vartheta^2 \wedge \vartheta^3 + F\,\vartheta^3 \wedge \vartheta^1,\end{aligned} \tag{4.42}$$

for some functions A, B, C, D, E, and F of (X, Y, Z). Cartan's first structural equations give us the following connection 1-forms

$$\omega^1{}_2 = A\vartheta^1 + C\vartheta^2, \quad \omega^2{}_3 = D\vartheta^2 + E\vartheta^3, \quad \omega^3{}_1 = B\vartheta^1 + F\vartheta^3. \quad (4.43)$$

From Cartan's second structural equations we obtain the following system of PDEs for flatness of the connection:

$$BD - A_{,Y} + C_{,X} = 0, \quad A_{,Z} - BE = 0, \quad C_{,Z} + FD = 0, \quad (4.44)$$
$$CF - D_{,Z} + E_{,Y} = 0, \quad D_{,X} - BC = 0, \quad E_{,X} + AF = 0, \quad (4.45)$$
$$AE + B_{,Z} - F_{,X} = 0, \quad F_{,Y} - AC = 0, \quad B_{,Y} + AD = 0. \quad (4.46)$$

We can now look at several cases. If $B = C = E = 0$, then there are two possible solutions

$$\mathcal{T}^1 = 0, \ \mathcal{T}^2 = D(Y)\, \vartheta^2 \wedge \vartheta^3, \ \mathcal{T}^3 = 0, \quad (4.47)$$
$$\mathcal{T}^1 = A(X)\, \vartheta^1 \wedge \vartheta^2, \ \mathcal{T}^2 = \mathcal{T}^3 = 0. \quad (4.48)$$

for arbitrary functions $A(X)$ and $D(Y)$.

If $A = D = F = 0$, then we have

$$\mathcal{T}^1 = 0, \ \ \mathcal{T}^2 = C(Y)\, \vartheta^1 \wedge \vartheta^2, \quad \mathcal{T}^3 = E(Z)\, \vartheta^2 \wedge \vartheta^3, \quad (4.49)$$

for arbitrary functions $C(Y)$ and $E(Z)$.

If $C = D = E = F = 0$, we have

$$\mathcal{T}^1 = A(X)\, \vartheta^1 \wedge \vartheta^2 + B(X)\, \vartheta^3 \wedge \vartheta^1, \ \ \mathcal{T}^2 = \mathcal{T}^3 = 0, \quad (4.50)$$

for arbitrary functions $A(X)$ and $B(X)$. Several other examples of zero-stress dislocation distributions can be similarly generated. It can be shown that if we take a combination of screw and edge dislocations, the screw dislocation part of the torsion 2-form always has to vanish for the dislocation distribution to be zero-stress.

Cylindrical coframe. Let us now look for zero-stress dislocation distributions in the following coframe field

$$\vartheta^1 = \mathrm{d}R, \quad \vartheta^2 = R\mathrm{d}\Phi, \quad \vartheta^3 = \mathrm{d}Z. \quad (4.51)$$

We know that the Levi-Civita connection of this metric is flat. Now, again if a given torsion distribution has a flat connection in this coframe field, then the given torsion distribution is zero stress. Let us first start with a distribution of screw distributions (not necessarily isotropic), that is

$$\mathcal{T}^1 = \xi\vartheta^2 \wedge \vartheta^3, \quad \mathcal{T}^2 = \eta\vartheta^3 \wedge \vartheta^1, \quad \mathcal{T}^3 = \lambda\vartheta^1 \wedge \vartheta^2, \quad (4.52)$$

for some functions ξ, η, and λ of (R, Φ, Z). Cartan's first structural equations give us the following connection 1-forms

$$\omega^1{}_2 = -\frac{1}{R}\vartheta^2 + f\vartheta^3, \quad \omega^2{}_3 = g\vartheta^1, \quad \omega^3{}_1 = h\vartheta^2, \quad (4.53)$$

where $f = \frac{-\xi-\eta+\lambda}{2}$, $g = \frac{\xi-\eta-\lambda}{2}$, and $h = \frac{-\xi+\eta-\lambda}{2}$. From Cartan's second structural equations we obtain the following system of PDEs for flatness of the connection:

$$f_{,R} = 0, \quad f_{,\Phi} = 0, \quad gh = 0, \tag{4.54}$$

$$g_{,\Phi} = 0, \quad g_{,Z} = 0, \quad fh = 0, \tag{4.55}$$

$$\frac{1}{R}(h-g) + h_{,R} = 0, \quad h_{,Z} = 0, \quad fg = 0. \tag{4.56}$$

It can be readily shown that all the solutions of this system are either

$$\mathcal{T}^1 = \frac{H(\Phi)}{R}\vartheta^2 \wedge \vartheta^3, \quad \mathcal{T}^2 = 0, \quad \mathcal{T}^3 = \frac{H(\Phi)}{R}\vartheta^1 \wedge \vartheta^2, \tag{4.57}$$

for arbitrary $H = H(\Phi)$, or

$$\mathcal{T}^1 = \xi(Z)\vartheta^2 \wedge \vartheta^3, \ \ \mathcal{T}^2 = \xi(Z)\vartheta^3 \wedge \vartheta^1, \ \ \mathcal{T}^3 = 0, \tag{4.58}$$

for arbitrary $\xi(Z)$.

Now let us look at edge dislocations and assume that torsion forms are given as

$$\begin{aligned}&\mathcal{T}^1 = A\,\vartheta^1 \wedge \vartheta^2 + B\,\vartheta^3 \wedge \vartheta^1, \quad \mathcal{T}^2 = C\,\vartheta^1 \wedge \vartheta^2 + D\,\vartheta^2 \wedge \vartheta^3,\\ &\mathcal{T}^3 = E\,\vartheta^2 \wedge \vartheta^3 + F\,\vartheta^3 \wedge \vartheta^1,\end{aligned} \tag{4.59}$$

for some functions A, B, C, D, E, and F of (R, Φ, Z). Cartan's first structural equations give us the following connection 1-forms

$$\omega^1{}_2 = A\vartheta^1 + \left(C - \frac{1}{R}\right)\vartheta^2, \quad \omega^2{}_3 = D\vartheta^2 + E\vartheta^3, \quad \omega^3{}_1 = B\vartheta^1 + F\vartheta^3. \tag{4.60}$$

From Cartan's second structural equations we obtain the following system of PDEs for flatness of the connection:

$$-\frac{1}{R}A_{,\Phi} + \frac{1}{R}C + C_{,R} + BD = 0, \quad C_{,Z} + DF = 0, \tag{4.61}$$

$$A_{,Z} - BE = 0, \quad \frac{1}{R}D + D_{,R} - B\left(C - \frac{1}{R}\right) = 0, \tag{4.62}$$

$$\frac{1}{R}E_{,\Phi} - D_{,Z} + F\left(C - \frac{1}{R}\right) = 0, \quad E_{,R} + AF = 0, \tag{4.63}$$

$$\frac{1}{R}B_{,\Phi} + AD = 0, \quad B_{,Z} - F_{,R} + AE = 0, \tag{4.64}$$

$$\frac{1}{R}F_{,\Phi} - E\left(C - \frac{1}{R}\right) = 0. \tag{4.65}$$

Choosing $B = E = F = 0$ these equations tell us that A, C, D, and D are functions of only R and Φ and

$$\frac{1}{R}A_{,\Phi} = \frac{1}{R}C + C_{,R}, \quad \frac{1}{R}D + D_{,R}, \quad AD = 0. \tag{4.66}$$

If $A = 0$, then we have the following solution

$$\mathcal{T}^1 = 0, \quad \mathcal{T}^2 = \frac{K(\Phi)}{R}\vartheta^1 \wedge \vartheta^2 + \frac{H(\Phi)}{R}\vartheta^2 \wedge \vartheta^3, \quad \mathcal{T}^3 = 0, \tag{4.67}$$

for arbitrary functions $K(\Phi)$ and $H(\Phi)$. If $D = 0$, then A and C are related by (4.66)$_3$. Choosing $C = 0$, we have the following solution:

$$\mathcal{T}^1 = A(R)\vartheta^1 \wedge \vartheta^2, \quad \mathcal{T}^2 = \mathcal{T}^3 = 0, \tag{4.68}$$

for arbitrary function $A(R)$.

4.3. Some Non-Trivial Zero-Stress Dislocation Distributions in Two Dimensions

We now describe a non-trivial example of zero-stress dislocation distributions in two dimensions. Let us start with the following orthonormal moving coframe field

$$\vartheta^1 = \mathrm{d}X, \quad \vartheta^2 = \mathrm{d}Y, \tag{4.69}$$

with metric $\mathbf{G} = \mathrm{d}X \otimes \mathrm{d}X + \mathrm{d}Y \otimes \mathrm{d}Y$. We know that the Levi-Civita connection of this metric is flat. Now, if a given torsion distribution has a flat connection in this coframe field, then the given torsion distribution is zero stress. In two dimensions, only edge dislocations are possible. We assume that

$$\mathcal{T}^1 = \xi(X, Y)\vartheta^1 \wedge \vartheta^1, \quad \mathcal{T}^2 = \eta(X, Y)\vartheta^1 \wedge \vartheta^2, \tag{4.70}$$

for some functions ξ and η of (X, Y). Cartan's first structural equation gives us the following connection 1-form

$$\omega^1{}_2 = \xi\vartheta^1 + \eta\vartheta^2. \tag{4.71}$$

From Cartan's second structural equation curvature 2-form is obtained as

$$\mathcal{R}^1{}_2 = \mathrm{d}\omega^1{}_2 = (-\xi_{,Y} + \eta_{,X})\vartheta^1 \wedge \vartheta^2. \tag{4.72}$$

Therefore, if $\xi_{,Y} = \eta_{,X}$ the edge dislocation distribution (4.70) is zero-stress.

4.4. Linearized Dislocation Mechanics

Let us start with a material manifold $(\mathcal{B}, \nabla, \mathbf{G})$. We assume that this manifold is flat at all times. A variation of dislocation density tensor (or the moving coframe field) would result in a variation of metric and also the deformation mapping. We would like to find the governing equations for the unknown deformation mapping variation or "displacement" field in the language of classical linearized elasticity. We can vary either the coframe field ϑ^α (and equivalently $\mathbf{F}_p$) or the metric. Note that given $(\delta F_p)^\alpha{}_A$, we have

$$\left(\delta F_p^{-1}\right)_\alpha{}^A = -\left(F_p^{-1}\right)_\beta{}^A \left(\delta F_p\right)^\beta{}_B \left(F_p^{-1}\right)_\alpha^B. \tag{4.73}$$

Therefore, variation of the torsion tensor reads

$$\delta T^A{}_{BC} = -\left(\delta F_p^{-1}\right)_\alpha{}^A\, T^M{}_{BC}\left(\delta F_p\right)^\alpha{}_M + \left(F_p^{-1}\right)_\alpha{}^A\left[\left(\delta F_p\right)^\alpha{}_{B,C} - \left(\delta F_p\right)^\alpha{}_{C,B}\right]. \tag{4.74}$$

Similarly, δG_{AB} can be calculated. In what follows, we assume that metric variation is given. Given an equilibrium configuration φ, balance of linear momentum reads

$$\operatorname{Div}\mathbf{P} + \rho_0\mathbf{B} = \mathbf{0}, \tag{4.75}$$

where $\mathbf{P}$ is the first Piola–Kirchhoff stress tensor and $\mathbf{B}$ is the body force. More precisely, φ is an equilibrium configuration with respect to the underlying material manifold $(\mathcal{B}, \mathbf{G})$. Now suppose that material metric changes to $\mathbf{G} + \delta\mathbf{G}$. Having a new material manifold, the equilibrium configuration changes. Here, we are interested in calculating $\delta\mathbf{P}$ for a given $\delta\mathbf{G}$ and then the governing equations for $\delta\varphi$.

For simplicity, we ignore body forces. To linearize (4.75) we need to simplify the following

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\left[P^{aA}(\varepsilon)_{|A}\right] \\
&= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\left[\frac{\partial P^{aA}(\varepsilon)}{\partial X^A} + \Gamma^A{}_{AB}(\varepsilon)P^{aB}(\varepsilon) + \gamma^a{}_{bc}F^b{}_A(\varepsilon)P^{cA}(\varepsilon)\right] \\
&= \frac{\partial}{\partial X^A}\delta P^{aA} + \Gamma^A{}_{AB}\delta P^{aB} + \delta\Gamma^A{}_{AB}P^{aB} + \gamma^a{}_{bc}\delta F^b{}_A P^{cA} + \gamma^a{}_{bc}F^b{}_A\delta P^{cA} \\
&= \delta P^{aA}{}_{|A} + \delta\Gamma^A{}_{AB}P^{aB} + \gamma^a{}_{bc}P^{cA}\delta F^b{}_A.
\end{aligned} \tag{4.76}$$

Let us denote $\mathbf{U} = \delta\varphi$ and note that [51] $\delta F^a{}_A = U^a{}_{|A}$. We also know that the variation of the Levi-Civita connection is given by (2.16). The first Piola–Kirchhoff stress is written as

$$\mathbf{P} = \rho_0\mathbf{g}^\sharp\frac{\partial\Psi}{\partial\mathbf{F}}, \tag{4.77}$$

where $\Psi = \Psi(\mathbf{X}, \Theta, \mathbf{G}, \mathbf{F}, \mathbf{g})$ is the material free energy density, Θ being the absolute temperature. Note that changing $\mathbf{G}$, the equilibrium configuration φ and hence $\mathbf{F}$ changes. Thus

$$\delta\mathbf{P} = \rho_0\mathbf{g}^\sharp\frac{\partial^2\Psi}{\partial\mathbf{F}\partial\mathbf{F}}:\delta\mathbf{F} + \rho_0\mathbf{g}^\sharp\frac{\partial^2\Psi}{\partial\mathbf{G}\partial\mathbf{F}}:\delta\mathbf{G} = \mathbb{A}:\delta\mathbf{F} + \mathbb{B}:\delta\mathbf{G}. \tag{4.78}$$

Hence

$$\operatorname{Div}(\delta\mathbf{P}) = \operatorname{Div}\left(\mathbb{A}:\delta\mathbf{F}\right) + \operatorname{Div}\left(\mathbb{B}:\delta\mathbf{G}\right). \tag{4.79}$$

Therefore, the linearized balance of linear momentum in component form reads

$$\begin{aligned}
&\left(\mathbb{A}^{aA}{}_b{}^B U^b{}_{|B}\right)_{|A} + \gamma^a{}_{bc}P^{cA}U^b{}_{|A} \\
&= -\frac{1}{2}G^{AD}\left(\delta G_{AD|B} + \delta G_{BD|A} - \delta G_{AB|D}\right)P^{aB} \\
&\quad - \left(\mathbb{B}^{MNaA}\delta G_{MN}\right)_{|A}.
\end{aligned} \tag{4.80}$$

Linearized impotency conditions. Let us linearize the nonlinear kinematics about the zero-dislocation distribution, that is, $\mathbf{F}_p = \mathbf{I}$. Consider a one-parameter family $\mathbf{F}_p(\varepsilon)$ such that

$$\mathbf{F}_p(0) = \mathbf{I}, \quad \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} \mathbf{F}_p(\varepsilon) = \boldsymbol{\beta}_p. \tag{4.81}$$

The material metric has the components $G_{AB}(\varepsilon) = \left(F_p(\varepsilon)\right)^\alpha{}_A \left(F_p(\varepsilon)\right)^\beta{}_B \delta_{\alpha\beta}$ and hence

$$\begin{aligned} \delta G_{AB} &= \left(\beta_p\right)^\alpha{}_A \delta^\beta_B \delta_{\alpha\beta} + \left(\beta_p\right)^\beta{}_B \delta^\alpha_A \delta_{\alpha\beta} \\ &= (\beta_p)_{AB} + (\beta_p)_{BA} \\ &= 2\left(\beta_p^{\mathsf{S}}\right)_{AB}. \end{aligned} \tag{4.82}$$

Assuming that $\mathcal{B}$ is simply-connected, a zero-stress dislocation distribution corresponds to $R_{AB} = 0$ or equivalently $\mathsf{E}_{AB} = 0$. It is known that in dimension three, the Einstein tensor has the following representation

$$\mathsf{E}^{AB} = \frac{1}{4}\varepsilon^{AMN}\varepsilon^{BPQ}\mathcal{R}_{MNPQ}, \tag{4.83}$$

where $\mathcal{R}_{MNPQ} = G_{PS}\mathcal{R}^S{}_{MNQ}$. The linearized Einstein tensor reads

$$\delta\mathsf{E}_{AB} = -\frac{1}{2}\varepsilon_{AMN}\varepsilon_{BPQ}\delta G_{MP,NQ} = -\varepsilon_{AMN}\varepsilon_{BPQ}\left(\beta_p^{\mathsf{S}}\right)_{MP,NQ}. \tag{4.84}$$

Note that[12]

$$\left(\mathsf{Curl}\circ\mathsf{Curl}\boldsymbol{\beta}_p^{\mathsf{S}}\right)_{AB} = \frac{1}{2}\varepsilon_{AMN}\varepsilon_{BPQ}\delta G_{MP,NQ}. \tag{4.85}$$

Therefore, $\delta\mathbf{E} = \mathbf{0}$ is equivalent to $\mathsf{Curl}\circ\mathsf{Curl}\boldsymbol{\beta}_p^{\mathsf{S}} = \mathbf{0}$. Similarly, it can be shown that the linearized impotency equations in terms of the dislocation density tensor read [65]

$$(\mathsf{Curl}\boldsymbol{\alpha})^{\mathsf{S}} = \mathbf{0}. \tag{4.86}$$

5. Continuum Mechanics of Solids with Distributed Dislocations

For an elastic solid with a distribution of dislocations, the material manifold (where by construction the body is stress-free) is a Weitzenböck manifold, that is, a flat metric-compatible manifold with torsion. Torsion of the material manifold can be obtained from the dislocation density tensor of classical dislocation mechanics. However, this manifold cannot be used directly to calculate the elastic energy. In nonlinear elasticity, stress, and consequently energy, depends on the changes of relative distances of material points. This means that we need a metric in the material

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[12] In components $(\mathsf{Curl}\boldsymbol{\xi})_{AB} = \varepsilon_{AMN}\partial_M\xi_{BN}$.

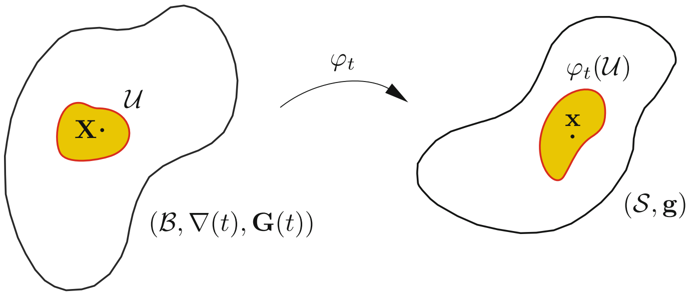

Fig. 4. Kinematic description of a continuum with distributed dislocations. Material manifold has evolving torsion and metric

manifold, that is, the metric compatible with the Weitzenböck connection. In terms of moving frames, having an orthonormal frame is equivalent to having the metric.

Given $\mathbf{F}_p$ we have a moving coframe and hence the metric. This metric is what Simo [67] denoted by $\mathbf{C}_p$, however without realizing that this is the metric of the rest configuration (he associated an arbitrary metric to his material manifold). If we assume that the distributed dislocation changes only the stress-free configuration of the body, energy depends on the dislocation distribution only through the material metric, that is, energy does not explicitly depend on the torsion tensor (dislocation density tensor), see [27] and [6]. However, dissipation can explicitly depend on torsion and its rate. In the case of bodies of grade-2, frame-indifference implies that energy explicitly depends on the dislocation density tensor, see [45]. In this paper, we work with simple bodies and our focus is on understanding the geometry of the material manifold, not its time evolution. In this section, we briefly explain how continuum balance laws can be derived covariantly but do not attempt to derive an evolution equation for the geometry of the material manifold.

A continuum with defects has local relaxed configurations that cannot be embedded in $\mathbb{R}^n$, that is, there are incompatibilities. However, one can embed the reference configuration in a non-Riemannian manifold with nonzero torsion and curvature. In a continuum with dislocations, in a deformation process dislocations evolve independently of the deformation mapping. At a given time t, dislocations have a new arrangement, that is, the incompatibility of the relaxed configuration is different from that of time $t = 0$. This means that at time t the reference configuration is $(\mathcal{B}, \nabla(t), \mathbf{G}(t))$, that is, defect evolution is represented by a time-dependent evolution of the connection and hence the metric. In other words, we think of connection and consequently metric as a dynamical variable of the field theory (note that $\nabla(t)\mathbf{G}(t) = \mathbf{0}$). Thus, in this field theory the ambient space is a Riemannian manifold $(\mathcal{S}, \mathbf{g})$ and motion is represented by the map $\varphi_t : \mathcal{B} \to \mathcal{S}$ and the time-dependent metric $\mathbf{G}(t)$. The kinematic description of motion is $(\mathcal{B}, \mathbf{G}(t)) \to (\mathcal{S}, \mathbf{g})$ as is shown in Fig. 4.

Let $\varphi : \mathcal{B} \to \mathcal{S}$ be a C^1 motion of $\mathcal{B}$ in $\mathcal{S}$. The material velocity field is defined by

$$\mathbf{V}(\mathbf{X}, t) = \frac{\partial}{\partial t}\varphi_t(\mathbf{X}). \tag{5.1}$$

The material acceleration is defined by

$$\mathbf{A}(\mathbf{X},t) = \frac{\partial \mathbf{V}(\mathbf{X},t)}{\partial t}. \tag{5.2}$$

In components, $A^a = \frac{\partial V^a}{\partial t} + \gamma^a{}_{bc} V^b V^c$, where $\gamma^a{}_{bc}$ are the coefficients of the Levi-Civita connection of $\mathbf{g}$ in the local coordinate chart $\{x^a\}$. The so-called deformation gradient is the tangent map of φ and is denoted by $\mathbf{F} = T\varphi$. Thus, at each point $\mathbf{X} \in \mathcal{B}$, it is a linear map

$$\mathbf{F}(\mathbf{X}) : T_{\mathbf{X}}\mathcal{B} \to T_{\varphi(\mathbf{X})}\mathcal{S}. \tag{5.3}$$

If $\{x^a\}$ and $\{X^A\}$ are local coordinate charts on $\mathcal{S}$ and $\mathcal{B}$, respectively, the components of $\mathbf{F}$ are

$$F^a{}_A(\mathbf{X}) = \frac{\partial \varphi^a}{\partial X^A}(\mathbf{X}). \tag{5.4}$$

Note that in this geometric formulation $\mathbf{F}$ is purely elastic; dislocations are represented by the evolving geometry of the material manifold. The transpose of $\mathbf{F}$ is defined as

$$\mathbf{F}^{\mathsf{T}} : T_{\mathbf{x}}\mathcal{S} \to T_{\mathbf{X}}\mathcal{B}, \quad \langle\!\langle \mathbf{F}\mathbf{V}, \mathbf{v} \rangle\!\rangle_{\mathbf{g}} = \left\langle\!\!\left\langle \mathbf{V}, \mathbf{F}^{\mathsf{T}}\mathbf{v} \right\rangle\!\!\right\rangle_{\mathbf{G}} \quad \forall\, \mathbf{V} \in T_{\mathbf{X}}\mathcal{B},\ \mathbf{v} \in T_{\mathbf{x}}\mathcal{S}. \tag{5.5}$$

In components, we have $(F^{\mathsf{T}}(\mathbf{X}))^A{}_a = g_{ab}(\mathbf{x}) F^b{}_B(\mathbf{X}) G^{AB}(\mathbf{X})$. The right Cauchy-Green deformation tensor is defined by

$$\mathbf{C}(X) : T_{\mathbf{X}}\mathcal{B} \to T_{\mathbf{X}}\mathcal{B}, \quad \mathbf{C}(\mathbf{X}) = \mathbf{F}^{\mathsf{T}}(\mathbf{X})\mathbf{F}(\mathbf{X}). \tag{5.6}$$

In components, $C^A_B = (F^{\mathsf{T}})^A{}_a F^a{}_B$. One can readily show that

$$\mathbf{C}^\flat = \varphi^*(\mathbf{g}) = \mathbf{F}^*\mathbf{g}\mathbf{F}, \text{ i.e. } \ C_{AB} = (g_{ab} \circ \varphi) F^a{}_A F^b{}_B. \tag{5.7}$$

In the geometric theory the following relation holds between volume elements of $(\mathcal{B}, \mathbf{G})$ and $(\mathcal{S}, \mathbf{g})$: $dv = J\, \mathrm{d}V$, where

$$J = \sqrt{\frac{\det \mathbf{g}}{\det \mathbf{G}}} \det \mathbf{F}. \tag{5.8}$$

An elastic deformation is isochoric if $J = 1$.

Remark 5.1. Note that for calculating residual stresses one needs to find a mapping from the underlying Riemannian material manifold to the ambient space (another Riemannian manifold). This means that all one needs is the material metric. In the case of dislocations one cannot calculate this metric directly; Cartan's structural equations need to be used.

Energy balance. Let us look at energy balance for a body with distributed dislocations. The standard material balance of energy for a subset $\mathcal{U} \subset \mathcal{B}$ reads [78]

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{U}} \rho_0 \left(E + \frac{1}{2} \langle\!\langle \mathbf{V}, \mathbf{V} \rangle\!\rangle_{\mathbf{g}} \right) \mathrm{d}V = \int_{\mathcal{U}} \rho_0 \left(\langle\!\langle \mathbf{B}, \mathbf{V} \rangle\!\rangle_{\mathbf{g}} + R \right) \mathrm{d}V + \int_{\partial \mathcal{U}} \left(\langle\!\langle \mathbf{T}, \mathbf{V} \rangle\!\rangle_{\mathbf{g}} + H \right) \mathrm{d}A, \tag{5.9}$$

where $E = E(\mathbf{X}, \mathsf{N}, \mathbf{G}, \mathbf{F}, \mathbf{g} \circ \varphi)$ is the material internal energy density, N, ρ_0, $\mathbf{B}$, $\mathbf{T}$, R, and H are specific entropy, material mass density, body force per unit undeformed mass, traction vector, heat supply, and heat flux, respectively.

YAVARI [83] showed that in the case of a growing body with a time-dependent material metric, energy balance should be modified. It was postulated that the term $\frac{\partial \mathbf{G}}{\partial t}$ should explicitly appear in the energy balance. In the case of an elastic body moving in a deforming ambient space, YAVARI and OZAKIN [84] proved that the term $\frac{\partial \mathbf{g}}{\partial t}$ should appear in the energy balance by first embedding the deforming ambient space in a larger and fixed manifold. Writing the standard energy balance they were able to reduce it to an energy balance written by an observer in the deforming ambient space.

We first note that the energy balance has to be modified in the case of a body with a time-dependent material metric. Note that when the metric is time dependent, the material density mass form $\mathsf{m}(\mathbf{X}, t) = \rho_0(\mathbf{X}, t)\mathrm{d}V(\mathbf{X}, t)$ is time dependent even if ρ_0 is not time dependent. For a sub-body $\mathcal{U} \subset \mathcal{B}$, conservation of mass reads

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{U}} \rho_0(\mathbf{X}, t)\mathrm{d}V(\mathbf{X}, t) = \int_{\mathcal{U}} \left[\frac{\partial \rho_0}{\partial t} + \frac{1}{2} \rho_0 \operatorname{tr}\left(\frac{\partial \mathbf{G}}{\partial t} \right) \right] \mathrm{d}V = 0. \tag{5.10}$$

In the case of a solid with distributed dislocations, the rate of change of material metric will then contribute to power. Therefore, energy balance for a dislocated body is postulated as

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{U}} \rho_0 \left(E + \frac{1}{2} \langle\!\langle \mathbf{V}, \mathbf{V} \rangle\!\rangle_{\mathbf{g}} \right) \mathrm{d}V = \int_{\mathcal{U}} \rho_0 \left(\langle\!\langle \mathbf{B}, \mathbf{V} \rangle\!\rangle_{\mathbf{g}} + R + \frac{\partial E}{\partial \mathbf{G}} : \frac{\partial \mathbf{G}}{\partial t} \right) \mathrm{d}V + \int_{\partial \mathcal{U}} \left(\langle\!\langle \mathbf{T}, \mathbf{V} \rangle\!\rangle_{\mathbf{g}} + H \right) \mathrm{d}A. \tag{5.11}$$

Remark 5.2. A changing ambient metric most likely comes from an actual moving constraint, for example a body restricted to move on a moving membrane (unless we are thinking about relativity, where the spacetime itself is dynamic, and dynamical metric need not come from some time-dependent embedding). But there is no physical reason to consider the change in material metric as coming from a time-dependent embedding. This is why we simply postulate the energy balance (5.11).

Covariance of energy balance. In continuum mechanics it is possible to derive all the balance laws using energy balance and its invariance under some groups of transformations. This was first discussed in [26] in the case of Euclidean ambient spaces and was extended to manifolds in [51]. See also [66,78–83] for applications of covariance ideas in different continuous and discrete systems.

In order to covariantly obtain all the balance laws, we postulate that energy balance is form invariant under an arbitrary time-dependent spatial diffeomorphism $\xi_t : \mathcal{S} \to \mathcal{S}$, that is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{U}}\rho_0'\left(E' + \frac{1}{2}\langle\!\langle \mathbf{V}', \mathbf{V}' \rangle\!\rangle_{\mathbf{g}'}\right)\mathrm{d}V \\ = \int_{\mathcal{U}}\rho_0'\left(\langle\!\langle \mathbf{B}', \mathbf{V}' \rangle\!\rangle_{\mathbf{g}'} + R' + \frac{\partial E'}{\partial \mathbf{G}'} : \frac{\partial \mathbf{G}'}{\partial t}\right)\mathrm{d}V \\ + \int_{\partial\mathcal{U}}\left(\langle\!\langle \mathbf{T}', \mathbf{V}' \rangle\!\rangle_{\mathbf{g}'} + H'\right)\mathrm{d}A.\end{aligned} \tag{5.12}$$

Note that [78] $R' = R,\ H' = H,\ \rho_0' = \rho_0,\ \mathbf{T}' = \xi_{t*}\mathbf{T},\ \mathbf{V}' = \xi_{t*}\mathbf{V} + \mathbf{W}$, where $\mathbf{W} = \frac{\partial}{\partial t}\xi_t \circ \varphi$. Also

$$\begin{aligned}&\mathbf{G}' = \mathbf{G}, \quad \frac{\partial \mathbf{G}'}{\partial t} = \frac{\partial \mathbf{G}}{\partial t}, \\ &E'(\mathbf{X}, \mathsf{N}', \mathbf{G}, \mathbf{F}', \mathbf{g} \circ \varphi') = E(\mathbf{X}, \mathsf{N}, \mathbf{G}, \mathbf{F}, \xi_t^*\mathbf{g} \circ \varphi).\end{aligned} \tag{5.13}$$

Therefore, at $t = t_0$

$$\frac{\mathrm{d}}{\mathrm{d}t}E' = \frac{\partial E}{\partial \mathsf{N}} : \dot{\mathsf{N}} + \frac{\partial E}{\partial \mathbf{G}} : \frac{\partial \mathbf{G}}{\partial t} + \frac{\partial E}{\partial \mathbf{g}} : \mathfrak{L}_{(\mathbf{V}+\mathbf{W})}\mathbf{g}, \tag{5.14}$$

where $\mathfrak{L}$ denotes the autonomous Lie derivative [51]. Body forces are assumed to transform such that [51] $\mathbf{B}' - \mathbf{A}' = \xi_{t*}(\mathbf{B} - \mathbf{A})$. Therefore, (5.12) at $t = t_0$ reads

$$\begin{aligned}\int_{\mathcal{U}}\left[\frac{\partial \rho_0}{\partial t} + \frac{1}{2}\rho_0 \operatorname{tr}\dot{\mathbf{G}}\right]\left(E + \frac{1}{2}\langle\!\langle \mathbf{V}+\mathbf{W}, \mathbf{V}+\mathbf{W} \rangle\!\rangle_{\mathbf{g}}\right)\mathrm{d}V \\ + \int_{\mathcal{U}}\rho_0\left(\frac{\partial E}{\partial \mathbf{G}} : \dot{\mathbf{G}} + \frac{\partial E}{\partial \mathbf{g}\circ\varphi} : \mathfrak{L}_{\mathbf{W}}\mathbf{g}\circ\varphi + \langle\!\langle \mathbf{V}+\mathbf{W}, \mathbf{A} \rangle\!\rangle_{\mathbf{g}}\right)\mathrm{d}V \\ = \int_{\mathcal{U}}\rho_0\left[\langle\!\langle \mathbf{B}, \mathbf{V}+\mathbf{W} \rangle\!\rangle_{\mathbf{g}} + R + \frac{\partial E}{\partial \mathbf{G}} : \dot{\mathbf{G}}\right]\mathrm{d}V \\ + \int_{\partial\mathcal{U}}\left(\langle\!\langle \mathbf{T}, \mathbf{V}+\mathbf{W} \rangle\!\rangle_{\mathbf{g}} + H\right)\mathrm{d}A.\end{aligned} \tag{5.15}$$

Subtracting (5.11) from (5.15), one obtains

$$\begin{aligned}\int_{\mathcal{U}}\left[\frac{\partial \rho_0}{\partial t} + \frac{1}{2}\rho_0 \operatorname{tr}\dot{\mathbf{G}}\right]\left(\frac{1}{2}\langle\!\langle \mathbf{W}, \mathbf{W} \rangle\!\rangle_{\mathbf{g}} + \langle\!\langle \mathbf{V}, \mathbf{W} \rangle\!\rangle_{\mathbf{g}}\right)\mathrm{d}V \\ + \int_{\mathcal{U}}\rho_0 \frac{\partial E}{\partial \mathbf{g}\circ\varphi} : \mathfrak{L}_{\mathbf{W}}\mathbf{g}\,\mathrm{d}V = \int_{\mathcal{U}}\rho_0\left(\langle\!\langle \mathbf{B}-\mathbf{A}, \mathbf{W} \rangle\!\rangle_{\mathbf{g}}\right)\mathrm{d}V \\ + \int_{\partial\mathcal{U}}\langle\!\langle \mathbf{T}, \mathbf{W} \rangle\!\rangle_{\mathbf{g}}\,\mathrm{d}A.\end{aligned} \tag{5.16}$$

Note that [78]

$$\int_{\partial\mathcal{U}} \langle\!\langle \mathbf{T}, \mathbf{W} \rangle\!\rangle_{\mathbf{g}} \, dA = \int_{\mathcal{U}} \left(\langle\!\langle \operatorname{Div} \mathbf{P}, \mathbf{W} \rangle\!\rangle_{\mathbf{g}} + \boldsymbol{\tau} : \boldsymbol{\Omega}^W + \frac{1}{2} \boldsymbol{\tau} : \mathfrak{L}_{\mathbf{W}} \mathbf{g} \right) dV, \tag{5.17}$$

where $\Omega^W_{ab} = \frac{1}{2}(W_{a|b} - W_{b|a})$, and $\boldsymbol{\tau} = J\boldsymbol{\sigma}$ is the Kirchhoff stress. From this and the arbitrariness of $\mathcal{U}$ and $\mathbf{W}$ we conclude that

$$\frac{\partial \rho_0}{\partial t} + \frac{1}{2} \rho_0 \operatorname{tr} \dot{\mathbf{G}} = 0, \tag{5.18}$$

$$\operatorname{Div} \mathbf{P} + \rho_0 \mathbf{B} = \rho_0 \mathbf{A}, \tag{5.19}$$

$$2\rho_0 \frac{\partial E}{\partial \mathbf{g}} = \boldsymbol{\tau}, \tag{5.20}$$

$$\boldsymbol{\tau}^{\mathsf{T}} = \boldsymbol{\tau}, \tag{5.21}$$

where $\mathbf{P}$ is the first Piola–Kirchhoff stress as before. Note that the divergence operator explicitly depends on $\mathbf{G}$, that is, the time dependency of material metric affects the governing balance equations. These governing equations are identical to those of an elastic body with bulk growth [83] with the only difference that mass is conserved.

Local form of energy balance. Note that

$$\frac{d}{dt} E = \mathbf{L}_V E = \frac{\partial E}{\partial \mathsf{N}} \dot{\mathsf{N}} + \frac{\partial E}{\partial \mathbf{G}} : \dot{\mathbf{G}} + \frac{\partial E}{\partial \mathbf{F}} : \mathbf{L}_{\mathbf{V}} \mathbf{F} + \frac{\partial E}{\partial \mathbf{g}} : \mathbf{L}_{\mathbf{V}} \mathbf{g} \circ \varphi, \tag{5.22}$$

where $\mathbf{L}$ denotes the Lie derivative [51]. We know that $\mathbf{L}_{\mathbf{V}} \mathbf{F} = \mathbf{0}$ because for an arbitrary (time-independent) $\mathbf{U} \in T_{\mathbf{X}} \mathcal{B}$

$$\begin{aligned}
\mathbf{L}_{\mathbf{V}}(\mathbf{F} \cdot \mathbf{U}) &= \left[\frac{d}{dt} \left(\psi^*_{t,s} \circ \varphi_{t*} \mathbf{U} \right) \right]_{s=t} \\
&= \left[\frac{d}{dt} \left[(\varphi_t \circ \varphi_s^{-1})^* \circ \varphi_{t*} \mathbf{U} \right] \right]_{s=t} \\
&= \left[\frac{d}{dt} \left(\varphi_{s*} \circ \varphi_t^* \circ \varphi_{t*} \mathbf{U} \right) \right]_{s=t} \\
&= \frac{d}{dt} \varphi_{s*} \mathbf{U} = \mathbf{0}.
\end{aligned} \tag{5.23}$$

Therefore

$$\frac{d}{dt} E = \frac{\partial E}{\partial \mathsf{N}} \dot{\mathsf{N}} + \frac{\partial E}{\partial \mathbf{G}} : \dot{\mathbf{G}} + \frac{\partial E}{\partial \mathbf{g}} : \mathbf{d}, \tag{5.24}$$

where $\mathbf{d} = \frac{1}{2} \mathfrak{L}_{\mathbf{V}} \mathbf{g} \circ \varphi$ is the rate of deformation tensor. We know that $H = -\left\langle\!\!\left\langle \mathbf{Q}, \hat{\mathbf{N}} \right\rangle\!\!\right\rangle_{\mathbf{G}}$, where $\mathbf{Q}$ is heat flux vector, and [78]

$$\int_{\partial\mathcal{U}} \langle\!\langle \mathbf{T}, \mathbf{V} \rangle\!\rangle_{\mathbf{g}} \, dA = \int_{\mathcal{U}} \left(\langle\!\langle \operatorname{Div} \mathbf{P}, \mathbf{V} \rangle\!\rangle_{\mathbf{g}} + \boldsymbol{\tau} : \boldsymbol{\Omega} + \boldsymbol{\tau} : \mathbf{d} \right) dV, \tag{5.25}$$

where $\Omega_{ab} = \frac{1}{2}(V_{a|b} - V_{b|a})$. Using balances of linear and angular momenta and (5.11) we obtain the local form of energy balance as

$$\rho_0 \frac{\mathrm{d}E}{\mathrm{d}t} + \operatorname{Div}\mathbf{Q} = \rho_0 \frac{\partial E}{\partial \mathbf{G}} : \dot{\mathbf{G}} + \boldsymbol{\tau} : \mathbf{d} + \rho_0 R. \tag{5.26}$$

In term of the first Piola–Kirchhoff stress, this can be written as

$$\rho_0 \frac{\mathrm{d}E}{\mathrm{d}t} + \operatorname{Div}\mathbf{Q} = \rho_0 \frac{\partial E}{\partial \mathbf{G}} : \dot{\mathbf{G}} + \mathbf{P} : \nabla_0 \mathbf{V} + \rho_0 R, \tag{5.27}$$

where $\mathbf{P} : \nabla_0 \mathbf{V} = P^{aA} V^a{}_{|A}$.

Dissipation. Consider a subbody $\mathcal{U} \subset \mathcal{B}$ and define dissipation as

$$\begin{aligned}\mathcal{D}(\mathcal{U}, t) = &\int_{\mathcal{U}} \rho_0 \langle\!\langle \mathbf{B}, \mathbf{V} \rangle\!\rangle_{\mathbf{g}} \,\mathrm{d}V + \int_{\partial\mathcal{U}} \langle\!\langle \mathbf{T}, \mathbf{V} \rangle\!\rangle_{\mathbf{g}} \,\mathrm{d}A \\ &- \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{U}} \rho_0 \left(E + \frac{1}{2} \langle\!\langle \mathbf{V}, \mathbf{V} \rangle\!\rangle_{\mathbf{g}} \right) \mathrm{d}V.\end{aligned} \tag{5.28}$$

Using conservation of mass, balance of linear and angular momenta, and the Doyle–Ericksen formula, dissipation is simplified to read

$$\mathcal{D}(\mathcal{U}, t) = - \int_{\mathcal{U}} \rho_0 \frac{\partial E}{\partial \mathbf{G}} : \dot{\mathbf{G}} \mathrm{d}V. \tag{5.29}$$

This is identical to SIMO's [67] dissipation (ignoring the internal state variables) and also very similar to GUPTA et al.'s [27] dissipation. Note that

$$\dot{G}_{AB} = (\dot{F}_p)^\alpha{}_A (F_p)^\beta{}_B \delta_{\alpha\beta} + (F_p)^\alpha{}_A (\dot{F}_p)^\beta{}_B \delta_{\alpha\beta}. \tag{5.30}$$

Therefore

$$\begin{aligned}\mathcal{D}(\mathcal{U}, t) &= -2 \int_{\mathcal{U}} \rho_0 \frac{\partial E}{\partial G_{AB}} (F_p)^\alpha{}_A (\dot{F}_p)^\beta{}_B \delta_{\alpha\beta} \mathrm{d}V \\ &= -2 \int_{\mathcal{U}} \rho_0 \frac{\partial E}{\partial \mathbf{G}} : \mathbf{F}_p^{\mathsf{T}} \dot{\mathbf{F}}_p \mathrm{d}V \\ &= 2 \int_{\mathcal{U}} \rho_0 \mathbf{G} \frac{\partial E}{\partial \mathbf{G}} : \dot{\mathbf{F}}_p^{-1} \mathbf{F}_p \mathrm{d}V.\end{aligned} \tag{5.31}$$

The second law of thermodynamics when the material metric is time dependent. For a classical nonlinear solid with a fixed material manifold, entropy production inequality in material coordinates reads [18]

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{U}} \rho_0 \mathsf{N} \mathrm{d}V \geqq \int_{\mathcal{U}} \frac{\rho_0 R}{\Theta} \mathrm{d}V + \int_{\partial\mathcal{U}} \frac{H}{\Theta} \,\mathrm{d}A, \tag{5.32}$$

where $\mathsf{N} = \mathsf{N}(\mathbf{X}, t)$ is the material entropy density and $\Theta = \Theta(\mathbf{X}, t)$ is the absolute temperature. When the material metric is time dependent, the extra dissipation (5.29) must be included in the Clausius–Duhem inequality (5.32) to read

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{U}} \rho_0 \mathsf{N} \mathrm{d}V \geqq \int_{\mathcal{U}} \frac{\rho_0 R}{\Theta} \mathrm{d}V + \int_{\partial\mathcal{U}} \frac{H}{\Theta} \,\mathrm{d}A + \int_{\mathcal{U}} \rho_0 \frac{\partial E}{\partial \mathbf{G}} : \dot{\mathbf{G}} \,\mathrm{d}V, \tag{5.33}$$

which is identical in form to that of solids with bulk growth [83]. This inequality can be localized to read

$$\rho_0 \frac{\mathsf{dN}}{\mathrm{d}t} \geqq \frac{\rho_0 R}{\Theta} - \operatorname{Div}\left(\frac{\mathbf{Q}}{\Theta}\right) + \rho_0 \frac{\partial E}{\partial \mathbf{G}} : \dot{\mathbf{G}}. \tag{5.34}$$

Following [18] and [51] we conclude that [83]

$$\frac{\partial \Psi}{\partial \Theta} = -\mathsf{N}, \quad \rho_0 \frac{\partial \Psi}{\partial \mathbf{F}} = \mathbf{P}, \tag{5.35}$$

and the entropy production inequality reduces to read

$$\rho_0 \frac{\partial \Psi}{\partial \mathbf{G}} : \dot{\mathbf{G}} + \frac{1}{\Theta} \mathbf{d}\Theta \cdot \mathbf{Q} \leqq 0. \tag{5.36}$$

6. Examples of Dislocated Solids, Their Material Manifolds, and Residual Stress Fields

In this section we look at several examples of nonlinear elastic bodies with single and distributed dislocations and obtain their material manifolds and residual stress fields. Most of the exact solutions presented in this section appear for the first time in the literature.

6.1. A Single Screw Dislocation

Rosakis and Rosakis [62] showed that for a single screw dislocation with a Burgers vector of magnitude b along the z-axis, the non-zero components of Cauchy stress are[13]:

$$\bar{\sigma}^{\phi z} = \bar{\sigma}^{z\phi} = \frac{\mu b}{2\pi} \frac{1}{r}, \quad \bar{\sigma}^{zz} = \frac{\mu b^2}{4\pi^2} \frac{1}{r^2}. \tag{6.1}$$

Note that the displacement field in the material and spatial cylindrical coordinates (R, Φ, Z) and (r, ϕ, z) has the following form:

$$(u_r, u_\phi, u_z) = \left(0, 0, -\frac{b\Phi}{2\pi}\right). \tag{6.2}$$

This means that

$$r = R, \quad \phi = \Phi, \quad z = Z + u(\Phi). \tag{6.3}$$

Acharya [3] using a coupled field theory with dislocation density as an independent field obtained the same stress distribution away from the dislocation core. Note that in curvilinear coordinates the components of a tensor may not have the same physical dimensions. The Cauchy stress components shown above are the so-called

13 See also [86] for similar solutions for both dislocations and disclinations with different constitutive assumptions.

physical components of Cauchy stress. We have emphasized this by putting a bar on the physical components. Note that the spatial metric in cylindrical coordinates has the form $\mathbf{g} = \mathrm{diag}(1, r^2, 1)$. The following relation holds between the Cauchy stress components (unbarred) and its physical components (barred) [70]

$$\bar{\sigma}^{ab} = \sigma^{ab}\sqrt{g_{aa}g_{bb}} \quad \text{no summation on } a \text{ or } b. \tag{6.4}$$

This means that for the nonzero Cauchy stress components we have

$$\sigma^{\phi z} = \sigma^{z\phi} = \frac{1}{r}\bar{\sigma}^{\phi z} = \frac{\mu b}{2\pi}\frac{1}{r^2}, \quad \sigma^{zz} = \bar{\sigma}^{zz} = \frac{\mu b^2}{4\pi^2}\frac{1}{r^2}. \tag{6.5}$$

The deformation gradient reads

$$\mathbf{F} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{b}{2\pi} & 1 \end{pmatrix}. \tag{6.6}$$

In cylindrical coordinates, the material metric has the form $\mathbf{G} = \mathrm{diag}(1, R^2, 1)$, hence in terms of physical components deformation gradient reads

$$\bar{\mathbf{F}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{b}{2\pi R} & 1 \end{pmatrix}. \tag{6.7}$$

In our geometric formulation we prefer to work with the unbarred components.

Material manifold of a screw dislocation. Here we study the material manifold of a body with a single screw dislocation along the Z-axis.[14] Let us denote the Euclidean 3-space by $\mathcal{B}_0$ with the flat metric

$$\mathrm{d}S^2 = \mathrm{d}R_0^2 + R_0^2\mathrm{d}\Phi_0^2 + \mathrm{d}Z_0^2, \tag{6.8}$$

in the cylindrical coordinates (R_0, Φ_0, Z_0). Now cut $\mathcal{B}_0$ along the half 2-plane Ω $(\Phi_0 = 0)$, and after translating by b in the Z_0-direction identify the two half 2-planes Ω^+ and Ω^- (see Fig. 5). We denote the identified manifold by $\mathcal{B}$. Note that in constructing $\mathcal{B}$ from $\mathcal{B}_0$, the Z-axis is removed. Note also that trajectories of the vector field $\partial/\partial\Phi_0$ are closed circles in $\mathcal{B}_0$. However, they fail to close in $\mathcal{B}$. The lack of closure is $-b$ and in the Z_0 direction. This means that $\mathcal{B}$ is flat everywhere but the Z-axis where there is a non-vanishing torsion. Following Tod [73] let us define the following smooth coordinates on $\mathcal{B}$.

$$R = R_0, \quad \Phi = \Phi_0, \quad Z = Z_0 - \frac{b}{2\pi}\Phi_0, \quad R_0 > 0. \tag{6.9}$$

In the new coordinate system the flat metric (6.8) has the following form

$$\mathrm{d}S^2 = \mathrm{d}R^2 + R^2\mathrm{d}\Phi^2 + \left(\mathrm{d}Z + \frac{b}{2\pi}\mathrm{d}\Phi\right)^2. \tag{6.10}$$

[14] Note that the material manifold depends only on the dislocation distribution and is independent of the constitutive equations of the body or any internal constraint, for example, incompressibility.

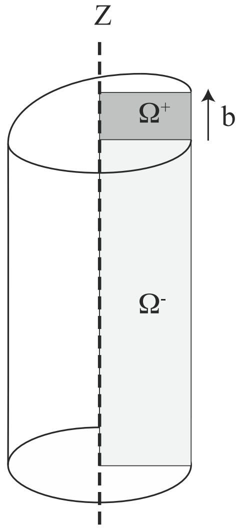

Fig. 5. Material manifold of a single screw dislocation. This is constructed using Volterra's cut-and-weld operation

Now our material metric and its inverse have the following forms

$$\mathbf{G} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & R^2 + \frac{b^2}{4\pi^2} & \frac{b}{2\pi} \\ 0 & \frac{b}{2\pi} & 1 \end{pmatrix}, \quad \mathbf{G}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{R^2} & \frac{b}{2\pi R^2} \\ 0 & -\frac{b}{2\pi R^2} & 1 + \frac{b^2}{4\pi^2 R^2} \end{pmatrix}. \tag{6.11}$$

Note that the dislocated solid is stress free in the material manifold $(\mathcal{B}, \mathbf{G})$ by construction. Now in the absence of external forces, we embed the body in the ambient space $(\mathcal{S}, \mathbf{g})$, which is the flat Euclidean 3-space. Because of symmetry of the problem we look for solutions of the form $(r, \phi, z) = (r(R), \Phi, Z)$. Note that by putting the dislocated body in the appropriate material manifold, the anelasticity problem is transformed to an elasticity problem mapping the material manifold with a non-trivial geometry to the Euclidean ambient space. Deformation gradient reads $\mathbf{F} = \mathrm{diag}(r'(R), 1, 1)$ and hence the incompressibility condition is written as

$$J = \sqrt{\frac{\det \mathbf{g}}{\det \mathbf{G}}} \det \mathbf{F} = \frac{r'(R) r(R)}{R} = 1. \tag{6.12}$$

Assuming that $r(0) = 0$ to fix the translation invariance, we get $r(R) = R$. For a neo-Hookean material we have

$$P^{aA} = \mu F^a{}_B G^{AB} - p \left(F^{-1}\right)_b{}^A g^{ab}, \tag{6.13}$$

where $p = p(R)$ is the pressure field. The non-zero stress components read

$$P^{rR} = \mu - p(R), \quad P^{\phi\Phi} = \frac{\mu - p(R)}{R^2}, \quad P^{\phi Z} = P^{z\Phi} = -\frac{\mu b}{2\pi R^2},$$
$$P^{zZ} = \mu - p(R) + \frac{\mu b^2}{4\pi^2 R^2}. \tag{6.14}$$

The only non-trivial equilibrium equation is ${P^{rA}}_{|A} = 0$, which is simplified to read $p'(R) = 0$, that is, the pressure field is uniform. The traction boundary condition at infinity implies that $p(R) = \mu$, hence we obtain the following non-zero stress components:

$$P^{\phi Z} = P^{z\Phi} = -\frac{\mu b}{2\pi R^2}, \quad P^{zZ} = \frac{\mu b^2}{4\pi^2 R^2}. \tag{6.15}$$

Having the first Piola–Kirchhoff stress, let us next find the Cauchy stress. We know that $\sigma^{ab} = \frac{1}{J} P^{aA} {F^b}_A$. Thus, the nonzero components of Cauchy stress are:

$$\begin{aligned} P^{\phi Z} &= \sigma^{\phi z} = -\frac{\mu b}{2\pi}\frac{1}{R^2}, \quad P^{z\Phi} = \sigma^{z\phi} = -\frac{\mu b}{2\pi}\frac{1}{R^2}, \\ P^{zZ} &= \sigma^{zz} = \frac{\mu b^2}{4\pi^2}\frac{1}{R^2}. \end{aligned} \tag{6.16}$$

Noting that $r = R$ we recover the result of [62]. See also [3].

Let us now use a generalized incompressible neo-Hookean material with the following form of strain energy density [49]

$$\mathcal{W} = \mathcal{W}(\mathbf{C}) = \frac{\mu}{2\alpha}(\text{tr}\,\mathbf{C})^{\alpha}, \tag{6.17}$$

where μ and α are material parameters. Now instead of (6.13) we have

$$P^{aA} = \mu(C_{MN}G^{MN})^{\alpha-1}{F^a}_B G^{AB} - p\left(F^{-1}\right)_b{}^A g^{ab}. \tag{6.18}$$

The non-zero stress components read[15]

$$\begin{aligned} P^{rR} &= \mu\left(3 + \frac{3b^2}{4\pi^2 R^2}\right)^{\alpha-1} - p(R), \quad P^{\phi\Phi} = \frac{1}{R^2}P^{rR}, \\ P^{\phi Z} &= P^{z\Phi} = -\frac{\mu b}{2\pi R^2}\left(3 + \frac{3b^2}{4\pi^2 R^2}\right)^{\alpha-1}, \\ P^{zZ} &= \mu\left(1 + \frac{b^2}{4\pi^2 R^2}\right)\left(3 + \frac{3b^2}{4\pi^2 R^2}\right)^{\alpha-1} - p(R). \end{aligned} \tag{6.19}$$

Equilibrium equations and the traction boundary conditions give the pressure field as

$$p(R) = \mu\left(3 + \frac{3b^2}{4\pi^2 R^2}\right)^{\alpha-1}. \tag{6.20}$$

15 Similar results are obtained in [62] for generalized neo-Hookean materials that are slightly different from the one we are using here.

Therefore, the non-zero stress components read

$$P^{\phi Z} = -\frac{\mu b}{2\pi}\frac{1}{R^2}\left(3+\frac{3b^2}{4\pi^2 R^2}\right)^{\alpha-1}, \quad P^{z\Phi} = -\frac{\mu b}{2\pi}\frac{1}{R^2}\left(3+\frac{3b^2}{4\pi^2 R^2}\right)^{\alpha-1},$$
$$P^{zZ} = \frac{\mu b^2}{4\pi^2}\frac{1}{R^2}\left(3+\frac{3b^2}{4\pi^2 R^2}\right)^{\alpha-1}. \tag{6.21}$$

Note that close to the dislocation line $P^{zZ} \sim \left(\frac{b^2}{4\pi^2}\right)^{\alpha} R^{-2\alpha}$, that is, stress singularity explicitly depends on the material parameter α. Cauchy stresses and their physical components read

$$\sigma^{\phi z} = \sigma^{z\phi} = -\frac{\mu b}{2\pi}\frac{1}{r^2}\left(3+\frac{3b^2}{4\pi^2 r^2}\right)^{\alpha-1},$$
$$\sigma^{zz} = \frac{\mu b^2}{4\pi^2}\frac{1}{r^2}\left(3+\frac{3b^2}{4\pi^2 r^2}\right)^{\alpha-1}, \tag{6.22}$$

and

$$\bar{\sigma}^{\phi z} = \bar{\sigma}^{z\phi} = -\frac{\mu b}{2\pi}\frac{1}{r^2}\left(3+\frac{3b^2}{4\pi^2 r^2}\right)^{\alpha-1},$$
$$\bar{\sigma}^{zz} = \frac{\mu b^2}{4\pi^2}\frac{1}{r^2}\left(3+\frac{3b^2}{4\pi^2 r^2}\right)^{\alpha-1}. \tag{6.23}$$

Remark 6.1. ZUBOV [86] noted that the resultant longitudinal force

$$2\pi \int_0^{R_0} P^{zZ}(R) R \mathrm{d}R, \tag{6.24}$$

is unbounded for the neo-Hookean material. We see that this force is finite for the generalized neo-Hookean material if $\alpha < 1$ (and for finite R_0).

Remark 6.2. Let us look at the material manifold of a single screw dislocation more carefully. Cartan's moving coframes read

$$\vartheta^1 = \mathrm{d}R, \quad \vartheta^2 = R\mathrm{d}\Phi, \quad \vartheta^3 = \mathrm{d}Z - \frac{b}{2\pi}\mathrm{d}\Phi. \tag{6.25}$$

The material metric is written as

$$G_{\alpha\beta} = \delta_{\alpha\beta}\vartheta^{\alpha} \otimes \vartheta^{\beta}. \tag{6.26}$$

We assume that in the material manifold torsion 2-forms are given as

$$T^1 = 0, \quad T^2 = 0, \quad T^3 = b\delta^2(R)\vartheta^1 \wedge \vartheta^2 = \frac{b\delta(R)}{2\pi}\mathrm{d}R \wedge \mathrm{d}\Phi. \tag{6.27}$$

To obtain the connection 1-forms we assume metric compatibility

$$\delta_{\alpha\gamma}\omega^{\gamma}{}_{\beta} + \delta_{\beta\gamma}\omega^{\gamma}{}_{\alpha} = 0, \tag{6.28}$$

and use Cartan's first structure equation

$$T^\alpha = \mathrm{d}\vartheta^\alpha + \omega^\alpha{}_\beta \wedge \vartheta^\beta. \tag{6.29}$$

If we now naively solve for connection 1-forms we will end up multiplying delta functions in calculating the curvature 1-forms. However, as we will see in the next example, the material manifold will be flat if we define $\vartheta^3 = \mathrm{d}Z + \frac{b}{2\pi}H(R)\mathrm{d}\Phi$. This shows that in the geometric framework single defects should be analyzed very carefully.

6.2. A Cylindrically-Symmetric Distribution of Parallel Screw Dislocations

Motivated by the first example, let us consider a cylindrically-symmetric distribution of screw dislocations parallel to the Z-axis (in a cylindrical coordinate system (R, Φ, Z)). Let us look for an orthonormal coframe field of the following form

$$\vartheta^1 = \mathrm{d}R, \quad \vartheta^2 = R\mathrm{d}\Phi, \quad \vartheta^3 = \mathrm{d}Z + f(R)\mathrm{d}\Phi, \tag{6.30}$$

for some unknown function f to be determined. Assuming metric compatibility, the unknown connection 1-forms are: $\omega^1{}_2, \omega^2{}_3, \omega^3{}_1$. For our distributed dislocation we assume the following torsion 2-forms[16]

$$\mathcal{T}^1 = \mathcal{T}^2 = 0, \quad \mathcal{T}^3 = \frac{R}{2\pi}b(R)\mathrm{d}R \wedge \mathrm{d}\Phi = \frac{b(R)}{2\pi}\vartheta^1 \wedge \vartheta^2, \tag{6.31}$$

where $b(R)$ is the radial density of the screw dislocation distribution. Note that

$$\begin{aligned} \mathrm{d}\vartheta^1 &= 0, \quad \mathrm{d}\vartheta^2 = \mathrm{d}R \wedge \mathrm{d}\Phi = \frac{1}{R}\vartheta^1 \wedge \vartheta^2, \\ \mathrm{d}\vartheta^3 &= f'(R)\mathrm{d}R \wedge \mathrm{d}\Phi = \frac{f'(R)}{R}\vartheta^1 \wedge \vartheta^2. \end{aligned} \tag{6.32}$$

[16] Note that

$$\begin{pmatrix} \vartheta^1 \\ \vartheta^2 \\ \vartheta^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & R & 0 \\ 0 & f(R) & 1 \end{pmatrix} \begin{pmatrix} \mathrm{d}R \\ \mathrm{d}\Phi \\ \mathrm{d}Z \end{pmatrix}.$$

In the coordinate frame we know that

$$T^1 = T^2 = 0, \quad T^3 = \frac{b(R)}{2\pi}\mathrm{d}R \wedge R\mathrm{d}\Phi = \frac{b(R)}{2\pi}\vartheta^1 \wedge \vartheta^2.$$

Therefore

$$\begin{pmatrix} \mathcal{T}^1 \\ \mathcal{T}^2 \\ \mathcal{T}^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & R & 0 \\ 0 & f(R) & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \frac{b(R)}{2\pi} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{b(R)}{2\pi} \end{pmatrix}.$$

Let us now use Cartan's first structural equations: $\mathcal{T}^\alpha = \mathrm{d}\vartheta^\alpha + \omega^\alpha{}_\beta \wedge \vartheta^\beta$. For $\alpha = 1, 2, 3$ these yield

$$\omega^1{}_{12} = \omega^3{}_{11} = 0, \quad \omega^3{}_{21} + \omega^1{}_{32} = 0, \tag{6.33}$$

$$\omega^1{}_{22} = -\frac{1}{R}, \quad \omega^2{}_{231} = 0, \quad \omega^1{}_{32} + \omega^2{}_{13} = 0, \tag{6.34}$$

$$\omega^3{}_{31} = \omega^2{}_{33} = 0, \quad \omega^3{}_{21} + \omega^2{}_{13} = \frac{f'(R)}{R} - \frac{b(R)}{2\pi}. \tag{6.35}$$

Therefore, the nonzero connection coefficients are

$$\omega^1{}_{22} = -\frac{1}{R}, \quad \omega^3{}_{21} = \omega^2{}_{13} = -\omega^1{}_{32} = \frac{f'(R)}{2R} - \frac{b(R)}{4\pi}. \tag{6.36}$$

Thus, the connection 1-forms read

$$\begin{aligned} \omega^1{}_2 &= -\frac{1}{R}\vartheta^2 - \left[\frac{f'(R)}{2R} - \frac{b(R)}{4\pi}\right]\vartheta^3, \quad \omega^2{}_3 = \left[\frac{f'(R)}{2R} - \frac{b(R)}{4\pi}\right]\vartheta^1, \\ \omega^3{}_1 &= \left[\frac{f'(R)}{2R} - \frac{b(R)}{4\pi}\right]\vartheta^2. \end{aligned} \tag{6.37}$$

We now enforce the material manifold to be flat, that is $\mathcal{R}^\alpha{}_\beta = \mathrm{d}\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma \wedge \omega^\gamma{}_\beta = 0$. Let us first look at $\mathcal{R}^2{}_3$. Denoting $h(R) = \frac{f'(R)}{2R} - \frac{b(R)}{4\pi}$, note that

$$\mathrm{d}\omega^2{}_3 = d(h(R)\vartheta^1) = h'(R)\mathrm{d}R \wedge \vartheta^1 = 0. \tag{6.38}$$

Thus

$$\mathcal{R}^2{}_3 = \omega^1{}_2 \wedge \omega^3{}_1 = h(R)^2\vartheta^2 \wedge \vartheta^3 = 0. \tag{6.39}$$

And therefore, $h(R) = 0$, that is

$$f'(R) = \frac{R}{2\pi}b(R). \tag{6.40}$$

It is easy to verify that $\mathcal{R}^1{}_2 = \mathcal{R}^3{}_1 = 0$ are trivially satisfied. Therefore, the material metric has the following form:

$$\mathbf{G} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & R^2 + f^2(R) & f(R) \\ 0 & f(R) & 1 \end{pmatrix}. \tag{6.41}$$

Note that $\det \mathbf{G} = 1$. Having the material manifold in order to obtain the residual stress field, we embed the material manifold into the ambient space, which is assumed to be the Euclidean three-space. We look for solutions of the form $(r, \phi, z) = (r(R), \Phi, Z)$, and hence $\det \mathbf{F} = r'(R)$. Assuming an incompressible neo-Hookean material, we have

$$J = \sqrt{\frac{\det \mathbf{g}}{\det \mathbf{G}}} \det \mathbf{F} = \frac{r}{R} r'(R) = 1. \tag{6.42}$$

Assuming that $r(0) = 0$ we obtain $r(R) = R$. For the neo-Hookean material we have $P^{aA} = \mu F^a{}_B G^{AB} - p\left(F^{-1}\right)_b{}^A g^{ab}$, where $p = p(R)$ is the pressure field. Thus, because $J = 1$ and $\mathbf{F} = \mathbf{I}$ we have

$$\mathbf{P} = \boldsymbol{\sigma} = \begin{pmatrix} \mu - p & 0 & 0 \\ 0 & \frac{\mu - p}{R^2} & -\mu\frac{f(R)}{R^2} \\ 0 & -\mu\frac{f(R)}{R^2} & (\mu - p) + \mu\frac{f(R)^2}{R^2} \end{pmatrix}. \tag{6.43}$$

Equilibrium equations $\operatorname{div}\boldsymbol{\sigma} = \mathbf{0}$ tell us that pressure field is uniform, that is, $p = p_0$. The traction boundary condition dictates $p_0 = \mu$. Therefore, the only nonzero stress components are

$$P^{\phi Z} = P^{z\Phi} = -\mu\frac{f(R)}{R^2}, \quad P^{zZ} = \mu\frac{f(R)^2}{R^2}. \tag{6.44}$$

Now if we instead use the constitutive equation (6.18) we would have the following stress components

$$P^{\phi Z} = P^{z\Phi} = -\mu\frac{f(R)}{R^2}\left(3 + \frac{f^2(R)}{R^2}\right)^{\alpha-1},$$
$$P^{zZ} = \mu\frac{f(R)^2}{R^2}\left(3 + \frac{f^2(R)}{R^2}\right)^{\alpha-1}. \tag{6.45}$$

Remark 6.3. For a single dislocation $b(R) = 2\pi b\delta^2(R)$, and hence

$$f'(R) = bR\delta^2(R) = \frac{b}{2\pi}\delta(R). \tag{6.46}$$

Therefore

$$f(R) = \frac{b}{2\pi}H(R) + C. \tag{6.47}$$

Note that $R > 0$ and for $b = 0$, $f = 0$ and hence $C = 0$, that is, $f(R) = b/2\pi$. For a uniform distribution, we have $b(R) = b_0$ and hence $f(R) = \frac{b_0}{4\pi}R^2$. Residual stresses in this case are:

$$P^{\phi Z} = P^{z\Phi} = -\frac{\mu b_0}{4\pi}, \quad P^{zZ} = \frac{\mu b_0^2}{16\pi^2}R^2. \tag{6.48}$$

Next, let us consider a few more examples. First, suppose that the distributed screw dislocations are supported on a cylinder of radius R_0 with its axis coincident with the Z-axis, that is $b(R) = 2\pi b_0\delta(C_{R_0})$. Note that

$$\delta(C_{R_0}) = \frac{1}{2\pi R}\delta(R - R_0) = \frac{1}{2\pi R_0}\delta(R - R_0). \tag{6.49}$$

Therefore, $f'(R) = \frac{b_0}{2\pi}\delta(R - R_0)$ and hence $f(R) = \frac{b_0}{2\pi}H(R - R_0)$. Hence, we have

$$P^{\phi Z} = P^{z\Phi} = -\frac{\mu b_0}{2\pi R^2}H(R - R_0), \quad P^{zZ} = \frac{\mu b_0^2}{4\pi^2 R^2}H(R - R_0). \tag{6.50}$$

It is seen that for $R < R_0$, stresses vanish and for $R > R_0$ stresses are identical to those of a single screw dislocation with Burgers vector b_0. The next example is what Acharya [3] considered, namely

$$b(R) = \frac{2b_0}{R_0}\left(\frac{1}{R} - \frac{1}{R_0}\right) H(R_0 - R) = \frac{2b_0}{R_0}\left(\frac{1}{R} - \frac{1}{R_0}\right)\left[1 - H(R - R_0)\right]. \tag{6.51}$$

It is easy to show that

$$f(R) = \frac{b_0}{\pi R_0}\left(R - \frac{R^2}{2R_0}\right) + \frac{b_0}{2\pi R_0^2}(R - R_0)^2 H(R - R_0). \tag{6.52}$$

Therefore, for $R < R_0$

$$P^{\phi Z} = P^{z\Phi} = -\frac{\mu b_0}{\pi R_0}\left(\frac{1}{R} - \frac{1}{2R_0}\right), \quad P^{zZ} = \frac{\mu b_0^2}{\pi^2 R_0^2}\left(1 - \frac{R}{2R_0}\right)^2, \tag{6.53}$$

and for $R > R_0$

$$P^{\phi Z} = P^{z\Phi} = -\frac{\mu b_0}{2\pi R^2}, \quad P^{zZ} = \frac{\mu b_0^2}{4\pi^2 R^2}. \tag{6.54}$$

These are identical to Acharya's [3] calculations. Note again that for $R > R_0$ stresses are identical to those of a single screw dislocation with Burgers vector b_0. Note also that traction vector is continuous on the surface $R = R_0$. Next, we show that this observation holds for a large class of distributed screw dislocations. We should mention that the following result is implicit in [3].

Proposition 6.4. *Suppose that we are given an arbitrary cylindrically-symmetric distribution of screw dislocations such that $b(R) = 0$ for $R > R_0$, where $b(R)$ is the density of the Burgers vectors and R_0 is some fixed radius. Also, define b_0 as*

$$\int_0^{R_0} \xi b(\xi) \mathrm{d}\xi = b_0. \tag{6.55}$$

Then, for $R > R_0$ stress distribution is independent of $b(R)$ and is identical to that of a single screw dislocation with Burgers vector b_0.

Proof. $f'(R) = \frac{R}{2\pi} b(R)$ and hence knowing that $f(R) = 0$ when $b(R) = 0$ we have

$$f(R) = \frac{1}{2\pi}\int_0^R \xi b(\xi) \mathrm{d}\xi. \tag{6.56}$$

For $R > R_0$:

$$f(R) = \frac{1}{2\pi}\int_0^{R_0} \xi b(\xi) \mathrm{d}\xi = \frac{b_0}{2\pi}. \tag{6.57}$$

Hence, for $R > R_0$:

$$P^{\phi Z}(R) = P^{z\Phi}(R) = -\frac{\mu b_0}{2\pi R^2}, \quad P^{zZ}(R) = \frac{\mu b_0^2}{4\pi^2 R^2}. \tag{6.58}$$

□

Now let us consider the following distributed screw dislocations: $b(R) = 2\pi b_0 \delta(C_{R_1}) - 2\pi b_0 \delta(C_{R_2})$, for $R_2 > R_1 > 0$. It can be readily shown that stresses are nonzero only in the cylindrical annulus $R_1 < R < R_2$, and there they are identical to those of a single screw dislocation with Burgers vector b_0. It should be noted that "principle" of superposition holds only for linear elasticity. However, in this particular example we see that the stress distribution of the distributed dislocation $2\pi b_0 \delta(C_{R_1}) - 2\pi b_0 \delta(C_{R_2})$ is the superposition of those of $2\pi b_0 \delta(C_{R_1})$ and $-2\pi b_0 \delta(C_{R_2})$.

6.3. An Isotropic Distribution of Screw Dislocations

Bloomer [12] constructed the material manifold of an isotropic distribution of screw dislocations, but did not calculate the corresponding stresses. Bloomer [12] started with the standard dislocation density tensor, that is, torsion of the connection is given. He then realized that the standard dislocation connection has zero curvature by construction. Having a torsion distribution, and knowing that curvature tensor vanishes, one is able to obtain the metric. He showed that the distributed screw dislocations must be uniform, otherwise the Ricci curvature tensor would not be symmetric. Here we repeat his calculations within our framework and will make an important observation at the end. Denoting the dislocation density by $M(\mathbf{X})$, torsion tensor would be completely anti-symmetric with the following form: $T_{ABC} = M\varepsilon_{ABC}$, where ε_{ABC} is the Levi-Civita tensor. Note that

$$\varepsilon_{MAB}\varepsilon^{M}{}_{CD} = G_{AC}G_{BD} - G_{AD}G_{BC}. \tag{6.59}$$

A simple calculation gives $K_{ABC} = \frac{M}{2}\varepsilon_{ABC}$. We need to calculate the material metric. Having torsion and contorsion tensors using the formula (2.22) and knowing that the curvature of the material manifold vanishes we can obtain the Ricci curvature tensor of the material manifold. Using (6.59) Ricci curvature is simplified to read

$$\overline{R}_{AB} = -\frac{1}{2}\frac{\partial M}{\partial X^C}\varepsilon^{C}{}_{AB} + \frac{M^2}{2}G_{AB}. \tag{6.60}$$

Symmetry of Ricci curvature implies that $\frac{1}{2}\frac{\partial M}{\partial X^A} = 0$ or $M(\mathbf{X}) = M_0$, that is, the screw dislocation distribution must be uniform otherwise the material connection is not metrizable[17]. Now the Riemannian curvature of the material manifold has the following form:

$$\overline{\mathcal{R}}_{ABCD} = \frac{M_0^2}{4}\left(G_{AC}G_{BD} - G_{AD}G_{BC}\right), \tag{6.61}$$

[17] This may mean that a non-uniform screw dislocation distribution induces point defects.

that is, material manifold has constant positive curvature. This is the 3-sphere with radius $R_0 = 2/M_0$. Note also that when there are no screw dislocations, that is, when $M_0 = 0$ the material manifold is the flat three-dimensional Euclidean space. The three-sphere cannot be embedded in the Euclidean three space. This immediately means that in the setting of classical nonlinear elasticity, that is, with no couple stresses, there is no solution. This is in agreement with Cartan's speculation [33].

This calculation can be checked using Cartan's moving frames. For a 3-sphere with Radius R_0 let us consider the hyperspherical coordinates (R, Ψ, Θ, Φ), for which the metric reads $(0 \leqq \Psi, \Theta \leqq \pi, 0 \leqq \Phi < 2\pi)$

$$dS^2 = R_0^2 d\Psi^2 + R_0^2 \sin^2 \Psi \left(d\theta^2 + \sin^2 \Theta d\Phi^2 \right). \tag{6.62}$$

We now choose the following orthonormal coframes

$$\vartheta^1 = R_0 d\Psi, \quad \vartheta^2 = R_0 \sin \Psi d\Theta, \quad \vartheta^3 = R_0 \sin \Psi \sin \Theta d\Phi. \tag{6.63}$$

Torsion 2-form has the following components

$$\mathcal{T}^1 = m_0 \vartheta^2 \wedge \vartheta^3, \ \mathcal{T}^2 = m_0 \vartheta^3 \wedge \vartheta^1, \ \mathcal{T}^3 = m_0 \vartheta^1 \wedge \vartheta^2. \tag{6.64}$$

Using Cartan's first structural equations $\mathcal{T}^\alpha = d\vartheta^\alpha + \omega^\alpha{}_\beta \wedge \vartheta^\beta$, the connection one-forms are obtained as

$$\begin{aligned} \omega^1{}_2 &= -\frac{cot \Psi}{R_0} \vartheta^2 - \frac{m_0}{2} \vartheta^3, \quad \omega^2{}_3 = -\frac{m_0}{2} \vartheta^1 - \frac{\cot \Theta}{R_0 \sin \Psi} \vartheta^3, \\ \omega^3{}_1 &= -\frac{m_0}{2} \vartheta^2 + \frac{\cot \Psi}{R_0} \vartheta^3. \end{aligned} \tag{6.65}$$

Using Cartan's second structural equations, the curvature 2-forms are obtained as

$$\begin{aligned} \mathcal{R}^1{}_2 &= \left(\frac{1}{R_0^2} - \frac{m_o^2}{4} \right) \vartheta^1 \wedge \vartheta^2, \quad \mathcal{R}^2{}_3 = \left(\frac{1}{R_0^2} - \frac{m_o^2}{4} \right) \vartheta^2 \wedge \vartheta^3, \\ \mathcal{R}^3{}_1 &= \left(\frac{1}{R_0^2} - \frac{m_o^2}{4} \right) \vartheta^3 \wedge \vartheta^1. \end{aligned} \tag{6.66}$$

The material manifold is flat if and only if

$$\frac{1}{R_0^2} = \frac{m_o^2}{4}. \tag{6.67}$$

Remark 6.5. The first Bianchi identity $d\mathcal{T}^\alpha + \omega^\alpha{}_\beta \mathcal{T}^\beta = \mathcal{R}^\alpha{}_\beta \wedge \vartheta^\beta = 0$ implies that $dm_0 = 0$, that is, the isotropic dislocation distribution must be uniform.

6.4. Edge Dislocation Distributions Uniform in Parallel Planes

Let us now consider a distribution of edge dislocations that are uniform in the planes parallel to the XY-plane but varying with Z. Let us consider the following moving coframe field

$$\vartheta^1 = \mathrm{e}^{\xi(Z)}\mathrm{d}X, \quad \vartheta^2 = \mathrm{e}^{\eta(Z)}\mathrm{d}Y, \quad \vartheta^3 = \mathrm{e}^{\lambda(Z)}\mathrm{d}Z. \tag{6.68}$$

This means that

$$\mathbf{e}_1 = \mathrm{e}^{-\xi(Z)}\partial_X, \quad \mathbf{e}_2 = \mathrm{e}^{-\eta(Z)}\partial_Y, \quad \mathbf{e}_3 = \mathrm{e}^{-\lambda(Z)}\partial_Z. \tag{6.69}$$

From (6.68) we obtain

$$\mathrm{d}\vartheta^1 = \xi'(Z)\mathrm{e}^{-\lambda(Z)}\vartheta^3 \wedge \vartheta^1, \quad \mathrm{d}\vartheta^2 = -\eta'(Z)\mathrm{e}^{-\lambda(Z)}\vartheta^2 \wedge \vartheta^3, \quad \mathrm{d}\vartheta^3 = 0. \tag{6.70}$$

We assume that the following torsion 2-forms are given

$$\mathcal{T}^1 = b(Z)\vartheta^3 \wedge \vartheta^1, \quad \mathcal{T}^2 = c(Z)\vartheta^2 \wedge \vartheta^3, \quad \mathcal{T}^3 = 0. \tag{6.71}$$

This represents a distribution of edge dislocations with Burgers vector

$$\mathbf{b} = \mathbf{b}(Z) = b(Z)\mathbf{e}_1 + c(Z)\mathbf{e}_2 = b(Z)\mathrm{e}^{-\xi(Z)}\partial_X + c(Z)\mathrm{e}^{-\eta(Z)}\partial_Y. \tag{6.72}$$

Using Cartan's first structural equations we obtain the following connection 1-forms:

$$\begin{aligned} &\omega^1{}_2 = 0, \quad \omega^2{}_3 = \left[c(Z) + \eta'(Z)\mathrm{e}^{-\lambda(Z)}\right]\vartheta^2, \\ &\omega^3{}_1 = \left[b(Z) - \xi'(Z)\mathrm{e}^{-\lambda(Z)}\right]\vartheta^1. \end{aligned} \tag{6.73}$$

If we assume that $\xi'(Z) = b(Z)\mathrm{e}^{\lambda(Z)}$ and $\eta'(Z) = -c(Z)\mathrm{e}^{\lambda(Z)}$, then from the second structural equations $\mathcal{R}^\alpha{}_\beta = 0$ is trivially satisfied. Let us now choose $\lambda(Z) = 0$. This implies then that $\xi'(Z) = b(Z)$ and $\eta'(Z) = -c(Z)$. Note that $\mathbf{G} = \mathrm{diag}(\mathrm{e}^{2\xi(Z)}, \mathrm{e}^{2\eta(Z)}, 1)$. We are looking for a solution of the form $(x, y, z) = (X + U(Z), Y + V(Z), Z)$. This then implies that $J = \mathrm{e}^{\xi(Z)+\eta(Z)}$. Incompressibility dictates that $\xi(Z) + \eta(Z) = 0$ and hence $\xi'(Z) + \eta'(Z) = 0$. This then means that $c(Z) = -b(Z)$. Now the first Piola–Kirchhoff stress tensor reads

$$\mathbf{P} = \begin{pmatrix} \mu\mathrm{e}^{-2\xi(Z)} - p(Z) & 0 & (\mu + p(Z))U'(Z) \\ 0 & \mu\mathrm{e}^{2\xi(Z)} - p(Z) & (\mu + p(Z))V'(Z) \\ 0 & 0 & \mu - p(Z) \end{pmatrix}. \tag{6.74}$$

Equilibrium equations are simpler to write for Cauchy stress, which reads

$$\boldsymbol{\sigma} = \begin{pmatrix} \mu\mathrm{e}^{-2\xi} - p + (p+\mu)U'^2 & (p+\mu)U'V' & (\mu+p)U' \\ (p+\mu)U'V' & \mu\mathrm{e}^{2\xi} - p + (p+\mu)V'^2 & (\mu+p)V' \\ (\mu-p)U' & (\mu-p)V' & \mu - p \end{pmatrix}. \tag{6.75}$$

Symmetry of Cauchy stress dictates $U'(Z) = V'(Z) = 0$, that is, up to a rigid translation in the XY-plane, $(x, y, z) = (X, Y, Z)$. Equilibrium equations dictate that $p'(Z) = 0$ and vanishing of traction vector on surfaces parallel to the X-Y plane gives us $p(Z) = \mu$. Therefore

$$\mathbf{P} = \mathrm{diag}\left\{\mu(\mathrm{e}^{-2\xi(Z)} - 1), \mu(\mathrm{e}^{2\xi(Z)} - 1), 0\right\}. \tag{6.76}$$

An interesting problem arises when the Burgers vector is given in the coordinate basis. Let us assume that

$$\mathbf{b} = b(Z)\partial_X = b(Z)\mathrm{e}^{\xi(Z)}\mathbf{e}_1. \tag{6.77}$$

Let us choose $\eta(Z) = 0$. Using Cartan's first structural equations we obtain the following connection 1-forms:

$$\omega^1{}_2 = 0, \quad \omega^2{}_3 = 0, \quad \omega^3{}_1 = \left[b(Z)\mathrm{e}^{\xi(Z)} - \xi'(Z)\mathrm{e}^{-\lambda(Z)}\right]\vartheta^1. \tag{6.78}$$

If we choose $b(Z)\mathrm{e}^{\xi(Z)} = \xi'(Z)\mathrm{e}^{-\lambda(Z)}$ the material connection will be flat as required for a distributed dislocation. We look for solutions of the form $(x, y, z) = (X+U(Z), Y+V(Z), Z)$. Note that $\mathbf{G} = \mathrm{diag}(\mathrm{e}^{2\xi(Z)}, 1, \mathrm{e}^{2\lambda(Z)})$. This then implies that $J = \mathrm{e}^{\xi(Z)+\lambda(Z)}$. Incompressibility dictates that $\xi(Z) + \lambda(Z) = 0$ and hence $\xi'(Z) = b(Z)$. For a neo-Hookean material the first Piola–Kirchhoff stress tensor reads

$$\mathbf{P} = \begin{pmatrix} \mu\mathrm{e}^{-2\xi(Z)} - p(Z) & 0 & (p(Z) + \mu\mathrm{e}^{2\xi})U'(Z) \\ 0 & \mu\mathrm{e}^{2\xi(Z)} - p(Z) & (p(Z) + \mu\mathrm{e}^{2\xi})V'(Z) \\ 0 & 0 & \mu\mathrm{e}^{2\xi} - p(Z) \end{pmatrix}. \tag{6.79}$$

Cauchy stress for the assumed displacement field reads

$$\boldsymbol{\sigma} = \begin{pmatrix} \mu\mathrm{e}^{-2\xi} - p + (p + \mu\mathrm{e}^{2\xi})U'^2 & (p + \mu\mathrm{e}^{2\xi})U'V' & (p + \mu\mathrm{e}^{2\xi})U' \\ (p + \mu\mathrm{e}^{2\xi})U'V' & \mu - p + (p + \mu\mathrm{e}^{2\xi})V'^2 & (p + \mu\mathrm{e}^{2\xi})V' \\ (-p + \mu\mathrm{e}^{2\xi})U' & (-p + \mu\mathrm{e}^{2\xi})V' & -p + \mu\mathrm{e}^{2\xi} \end{pmatrix}. \tag{6.80}$$

Symmetry of Cauchy stress dictates $U'(Z) = V'(Z) = 0$, that is again up to a rigid translation in the XY-plane, $(x, y, z) = (X, Y, Z)$. Equilibrium equations tell us that $\left(-p + \mu\mathrm{e}^{2\xi}\right)' = 0$ and hence (knowing that for $b = 0$, $\xi = 0$) this gives us $p = \mu\mathrm{e}^{2\xi}$. Therefore

$$\mathbf{P} = \mathrm{diag}\left\{\mu(\mathrm{e}^{-2\xi(Z)} - \mathrm{e}^{2\xi(Z)}), \mu(1 - \mathrm{e}^{2\xi(Z)}), 0\right\}. \tag{6.81}$$

6.5. Radially-Symmetric Distribution of Edge Dislocations in a Disk

Let us consider a flat disk with a distribution of edge dislocations with radial Burgers vectors. We assume that the flat disk is forced to remain flat, that is, we are looking at a two-dimensional problem. With respect to the polar coordinates (R, Φ), the torsion 2-form is assumed to have the following components in the coordinate frame:

$$T^1 = \frac{b(R)}{2\pi} \mathrm{d}R \wedge R\mathrm{d}\Phi, \quad T^2 = 0. \tag{6.82}$$

Let us choose the following orthonormal coframe field

$$\vartheta^1 = \mathrm{d}R + f(R)\mathrm{d}\Phi, \quad \vartheta^2 = R\mathrm{d}\Phi, \tag{6.83}$$

for a function f to be determined. Note that $\mathrm{d}R \wedge R\mathrm{d}\Phi = \vartheta^1 \wedge \vartheta^2$. The transformation F has the following form

$$\begin{pmatrix} \vartheta^1 \\ \vartheta^2 \end{pmatrix} = \begin{pmatrix} 1 & f(R) \\ 0 & R \end{pmatrix} \begin{pmatrix} \mathrm{d}R \\ \mathrm{d}\Phi \end{pmatrix}. \tag{6.84}$$

Therefore

$$\begin{pmatrix} \mathcal{T}^1 \\ \mathcal{T}^2 \end{pmatrix} = \begin{pmatrix} 1 & f(R) \\ 0 & R \end{pmatrix} \begin{pmatrix} \frac{b(R)}{2\pi} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{b(R)}{2\pi} \\ 0 \end{pmatrix}. \tag{6.85}$$

Using Cartan's first structural equations we have

$$\begin{aligned} \mathcal{T}^1 &= \mathrm{d}\vartheta^1 + \omega^1{}_2 \wedge \vartheta^2 = \left(\frac{f'(R)}{R} + \omega^1{}_{12} \right) \vartheta^1 \wedge \vartheta^2, \\ \mathcal{T}^2 &= \mathrm{d}\vartheta^2 - \omega^1{}_2 \wedge \vartheta^1 = \left(\frac{1}{R} + \omega^1{}_{22} \right) \vartheta^1 \wedge \vartheta^2. \end{aligned} \tag{6.86}$$

Therefore, the connection 1-form is obtained as

$$\omega^1{}_2 = h(R)\vartheta^1 - \frac{1}{R}\vartheta^2, \tag{6.87}$$

where $h(R) = \frac{b(R)}{2\pi} - \frac{f'(R)}{R}$. Using Cartan's second structural equation we have

$$\mathcal{R}^1{}_2 = \mathrm{d}\omega^1{}_2 = \frac{[f(R)h(R)]'}{R} \vartheta^1 \wedge \vartheta^2 = 0. \tag{6.88}$$

Therefore, $f(R)h(R) = C$, where C is a constant. But we know that when $b(R) = 0$, $f(R) = 0$ and hence $h(R) = 0$, that is, $C = 0$. We also know that for a nonvanishing $b(R)$, $f(R) \neq 0$. Thus, $h(R) = 0$, that is

$$f'(R) = \frac{R}{2\pi} b(R). \tag{6.89}$$

The material metric and its inverse have the following non-diagonal forms

$$\mathbf{G} = \begin{pmatrix} 1 & f(R) \\ f(R) & R^2 + f^2(R) \end{pmatrix}, \quad \mathbf{G}^{-1} = \begin{pmatrix} 1 + \frac{f^2(R)}{R^2} & -\frac{f(R)}{R^2} \\ -\frac{f(R)}{R^2} & \frac{1}{R^2} \end{pmatrix}. \tag{6.90}$$

Spatial metric reads $\mathbf{g} = \mathrm{diag}(1, r^2)$. We are now looking for solutions of the form $(r, \phi) = (r(R), \Phi)$. Assuming that the body is incompressible and $r(0) = 0$, we obtain $r(R) = R$. For a neo-Hookean material

$$\mathbf{P} = \boldsymbol{\sigma} = \begin{pmatrix} \mu - p(R) + \mu \frac{f^2(R)}{R^2} & -\mu \frac{f(R)}{R^2} \\ -\mu \frac{f(R)}{R^2} & \frac{\mu - p(R)}{R^2} \end{pmatrix}, \tag{6.91}$$

where $p(R)$ is the pressure field. The only non-trivial equilibrium equitation in terms of Cauchy stress reads

$$\frac{\partial \sigma^{rr}}{\partial r} + \frac{1}{r}\sigma^{rr} - r\sigma^{\phi\phi} = 0. \tag{6.92}$$

This gives

$$p'(R) = \frac{\mu f(R) b(R)}{\pi R} - \frac{\mu f^2(R)}{R^3}. \tag{6.93}$$

We know that when $b(R) = 0$, $p(R) = \mu$. Thus

$$p(R) = \mu + \mu \int_0^R \left[\frac{f(\xi) b(\xi)}{\pi \xi} - \frac{f^2(\xi)}{\xi^3} \right] \mathrm{d}\xi. \tag{6.94}$$

Therefore

$$\sigma^{rr} = \mu \frac{f^2(R)}{R^2} - \mu \int_0^R \left[\frac{f(\xi) b(\xi)}{\pi \xi} - \frac{f^2(\xi)}{\xi^3} \right] \mathrm{d}\xi, \quad \sigma^{r\phi} = -\mu \frac{f(R)}{R^2},$$
$$\sigma^{\phi\phi} = -\frac{\mu}{R^2} \int_0^R \left[\frac{f(\xi) b(\xi)}{\pi \xi} - \frac{f^2(\xi)}{\xi^3} \right] \mathrm{d}\xi. \tag{6.95}$$

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