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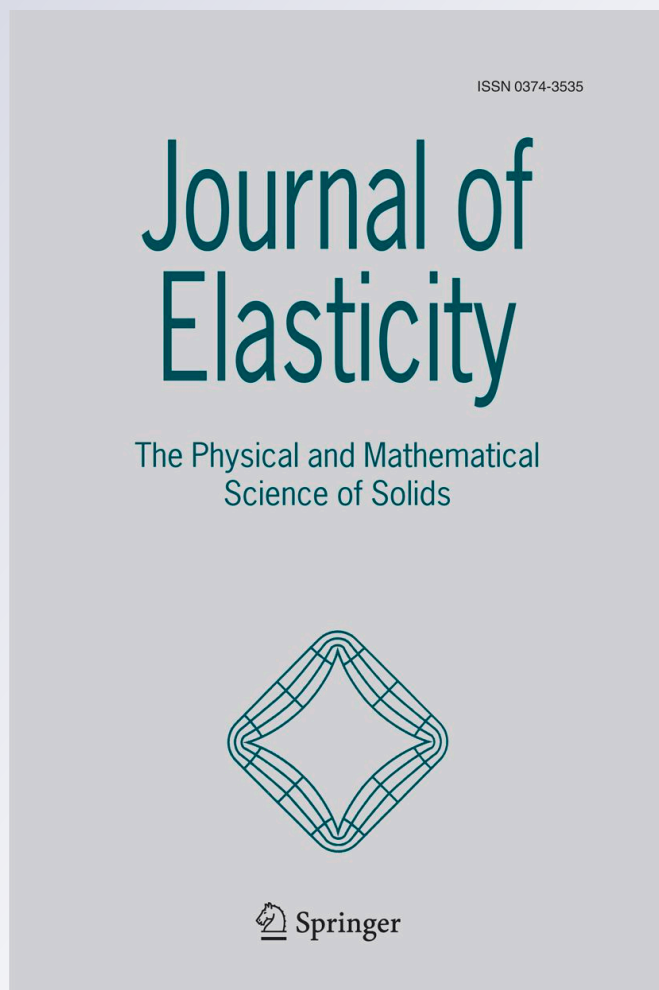
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Stability Estimates for a Twisted Rod Under Terminal Loads: A Three-dimensional Study

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Abstract The stability of an inextensible unshearable elastic rod with quadratic strain energy density subject to end loads is considered. We study the second variation of the corresponding rod-energy, making a distinction between in-plane and out-of-plane perturbations and isotropic and anisotropic cross-sections, respectively. In all cases, we demonstrate that the naturally straight state is a local energy minimizer in parameter regimes specified by material constants. These stability results are also accompanied by instability results in parameter regimes defined in terms of material constants.

Keywords Cosserat rods · Stability of straight axis solution · Variational criterion

Mathematics Subject Classification (2000) 74B20 · 49K40

1 Introduction

The buckling of rods under loads is the archetypical stability problem in continuum mechanics, dynamical systems, and bifurcation theory. First described by Euler for the case of a beam under compressive force [10, 11], the general problem of stability of elastic structures under loads and constraints is a major topic in continuum mechanics of great technological importance [4]. There are essentially three methods to approach the problem of stability in rods. First, we have the method of static bifurcation as originally used by Michell [15, 23] and Timoshenko [27]. This method consists in identifying values of a parameter (the control parameter) for which the linearized system (around a reference state) admits a non-trivial solution. This is accompanied by the assumption that the reference state of the system loses stability with respect to a bifurcating branch at the critical control parameter. This method can be easily applied to many systems with great success, leading to a simple and elegant way for developing some insight into possible behaviors of rods. However, it fails both in providing rigorous results (there is, in general, no basis for the assumed stability of the new

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branch) and in handling more complicated situations (for instance the case of a follower tangential load on a rod proposed by Beck [5] cannot be properly analyzed by a static bifurcation study). For these non-conservative problems, the dynamical bifurcation method is more appropriate. Within the framework of this second method, one studies the time evolution of small perturbations around both the trivial and bifurcated solutions for the linearized problem [1, 17, 29]. The spectrum of the linearized operator provides information about the exponential instability of some of these solutions which, for some prescribed dynamics, can yield rigorous results [6]. The third approach is a variational approach whereby we describe stable equilibria as local minimizers of a suitably defined elastic energy [7]. While this method is only valid for problems including conservative loads, it provides rigorous estimates for both the stability and instability of rods under loads. Furthermore, there exist a large literature on this type of problems providing necessary and sufficient criteria for stability in terms of the second variation of the elastic energy [21], e.g., Jacobi's condition, Legendre's condition and these have been successfully applied to a large class of nonlinear rods [22].

While it is usually the trend to generalize methods and systems, we run the risk of not seeing the tree for the forest. Here, we present a self-contained proof for the stability of a naturally straight, inextensible, unshearable rod in the (x, y) -plane, with a quadratic strain-energy density, under tension and controlled end-rotation. This problem is of great interest in the study of single-molecule DNA experiments [14]. As a stability problem, similar systems were first studied by Greenhill [18] and revisited by many authors through the method of static bifurcation or linear stability analysis [29]. We address the question of stability by studying the second variation of the strain energy directly, i.e., without appealing to either Legendre or Jacobi conditions or conjugate point methods. We formulate the stability problem in terms of Euler angles (θ, ϕ, ψ) (similar examples can be found in [21]) and by means of a direct computation, show that the energy density is strictly convex with respect to the first derivatives provided that the polar singularities, $\theta = 0$ and $\theta = \pi$, are not encountered. The strict convexity allows us to infer the local energy minimality of the naturally straight state from the positivity of the second variation [3, 19]. We use integral inequalities, e.g., Wirtinger's inequality, to obtain explicit upper and lower bounds for the critical tension F_c in terms of the material parameters, rod-length and end-rotation such that the rod is naturally straight for $F \geq F_c$ and deforms for $F < F_c$. For the case of a rod with an isotropic cross-section and subject to in-plane perturbations, these bounds match and yield an exact formula for the critical tension, which is simply the classical buckling result [26]. When we consider the fully three-dimensional problem with out-of-plane perturbations, these bounds do not match as they also depend on the magnitude of the controlled end-rotation, referred to as *twist* in the rest of the paper. We show that these methods can be readily generalized to rods with anisotropic cross-sections and the picture is qualitatively unchanged. The paper is organized as follows. We recall the main constituents of the elastic model in Sect. 2. In Sect. 3, we derive the governing equilibrium Euler-Lagrange equations and in Sects. 4.1 and 4.2, we analyze the stability of a naturally straight rod with an anisotropic and isotropic cross-section respectively, in terms of the second variation of the strain energy. We summarize our main conclusions in Sect. 5.

2 Preliminaries

Consider a naturally straight, uniform, inextensible and unshearable elastic rod. We work with a *Kirchhoff rod* under the assumptions of hyper-elasticity (the existence of a strain

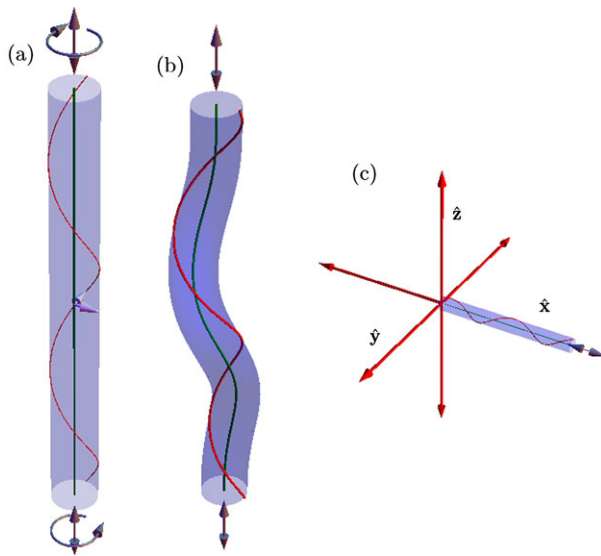


Fig. 1 A depiction of the type of rod geometry considered. The equilibrium configuration is shown in (a). The applied loads $F\hat{x}$ are represented by bi-directional arrows, to indicate tension or compression. A moment pair used to apply a left-handed end-rotation M is also shown. A curve drawn out by the vector $\mathbf{d}_1(s)$ is shown on the surface, indicating the uniform twist of the equilibrium solution we study. An example of the type of perturbation considered is shown in (b). The tangent vector \mathbf{d}_3 points along \hat{x} at the end points and the force is considered to act at the rod's ends along \hat{x} . The centerline of the equilibrium configuration is aligned with the \hat{x} axis, as shown in (c)

energy density) and linear constitutive relations between moments and strains of the rod [2, 20, 21, 24].

The elastic rod is initially aligned along the \hat{x} axis of a Cartesian basis $\{\hat{x}, \hat{y}, \hat{z}\}$. The rod is of fixed length L , is clamped at both ends and subject to a controlled end-rotation characterized by a non-zero parameter M , referred to as the twist parameter in the paper. An external force F , in the x -direction, is applied to both ends (see Figs. 1(a) and 1(c)). We parameterize the rod by an arc-length variable, s , where $s \in [0, L]$. The state of the rod is specified by its center-line, $\mathbf{r}(s) : [0, L] \rightarrow \mathbb{R}^3$, and a triad of orthogonal vectors known as *directors*: $\{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ where $\mathbf{d}_3 = \frac{d\mathbf{r}}{ds}$ is the local tangent to the rod [21]. Collectively, the directors define a right-handed rod-centered co-ordinate frame that relates the stressed and unstressed states of the rod. The director evolution along the rod is governed by the following system of ordinary differential equations:

$$\mathbf{d}'_i = \mathbf{u} \times \mathbf{d}_i, \quad i = 1, 2, 3 \tag{1}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ is the strain vector. We note that \mathbf{d}'_i denotes the derivative of \mathbf{d}_i with respect to s and this notation will be followed throughout the paper. The strains u_1 and u_2 are related to the geometric curvature (κ) of the rod-axis as $\kappa = \sqrt{u_1^2 + u_2^2}$. The function u_3 measures physical twist [21] (see Figs. 1(a) and 1(b)). It is composed of the geometric torsion τ of the curve's axis and an additional rotation ν of the material body which is independent of the axial geometry. This additional rotation ν is important here as the straight axis state has no torsion and the twist rate in this state is ν' . To be clear, we can relate these

quantities to the strains u_i as

$$u_1 = \kappa \sin(v), \quad u_2 = \kappa \cos(v), \quad u_3 = \tau + v' \tag{2}$$

see, e.g., Love [20] (section 253 with $v = f$).

We close the system by introducing a strain energy density function $W(u_1, u_2, u_3)$ that is quadratic in the strain components [8, 9],

$$W(u_1, u_2, u_3) = \frac{1}{2} u_i M_{ij} u_j, \quad i, j = 1, 2, 3, \tag{3}$$

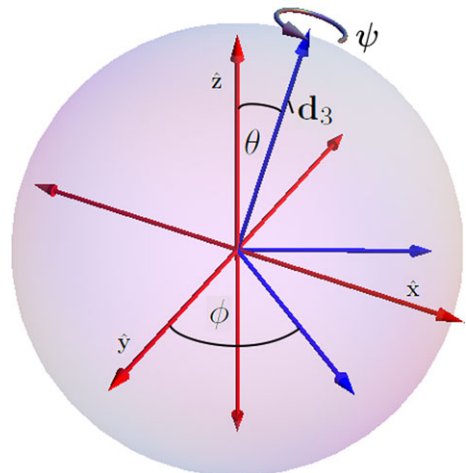
where we have used Einstein summation convention and $\mathbf{M} \in \mathbb{M}^{3 \times 3}$ is a constant, positive-definite symmetric 3×3 matrix whose components can be related to the elastic constants of the rod. In particular, the positive-definiteness requires that the diagonal elements, $\{M_{11}, M_{22}, M_{33}\}$, are positive where M_{11}, M_{22} are the bending rigidities and M_{33} is the torsional rigidity respectively [21].

In what follows, we introduce a set of Euler angles $\{\theta(s), \phi(s), \psi(s)\} \in C^1([0, L]; \mathbb{R})$ (C^1 denotes the space of continuously differentiable functions) to describe the orientation of the directors $\{\mathbf{d}_i(s)\}$ with respect to the unstressed state. In what follows we shall denote a particular configuration by a triplet of Euler angles with $\Theta = \{\theta, \psi, \phi\}$. We use the Euler angle representation detailed in Antman [2] (p. 320) and this mapping is defined by the tensor operation

$$\begin{aligned} & \begin{pmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \\ \mathbf{d}_3 \end{pmatrix} \\ &= \begin{pmatrix} -\sin \psi \sin \phi + \cos \psi \cos \phi \cos \theta, & \sin \psi \cos \phi + \cos \psi \sin \phi \cos \theta, & -\cos \psi \sin \theta \\ -\cos \psi \sin \phi - \sin \psi \cos \phi \cos \theta, & \cos \psi \cos \phi - \sin \psi \sin \phi \cos \theta, & \sin \psi \sin \theta \\ \sin \theta \cos \phi, & \sin \theta \sin \phi, & \cos \theta \end{pmatrix} \\ &\times \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix}, \tag{4} \end{aligned}$$

see Fig. 2. The strain components are given in terms of the Euler angles by

Fig. 2 A visualization of the Euler angle rotations $\{\theta, \phi, \psi\}$, used to map the Cartesian basis $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$ to the director basis $\{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$. First the Cartesian basis is rotated through an angle ϕ about $\hat{\mathbf{z}}$. A rotation of angle θ about the image of the $\hat{\mathbf{y}}$ vector is then applied. This maps the vector $\hat{\mathbf{z}}$ to \mathbf{d}_3 . Finally a rotation of angle ψ is applied about \mathbf{d}_3 in order to obtain the director basis



$$\begin{aligned}
 u_1 &= -\phi' \sin \theta \cos \psi + \theta' \sin \psi, \\
 u_2 &= \phi' \sin \theta \sin \psi + \theta' \cos \psi, \\
 u_3 &= \phi' \cos \theta + \psi'.
 \end{aligned}
 \tag{5}$$

This particular representation has a singularity at $\theta = 0, \pi$, at which the angles ϕ and ψ are not defined. Our work focuses on the stability of a naturally straight elastic rod. The polar singularity is an artefact of the coordinate system and hence, we work with a rod aligned along the x -axis with the external forces applied in the x -direction; the x -direction corresponds to a unique definition of the Euler angles: $\theta = \frac{\pi}{2}, \phi = \psi = 0$. The clamped boundary conditions translate into a Dirichlet boundary-value problem for $\theta(s), \phi(s) : [0, L] \rightarrow \mathbb{R}$ given by

$$\begin{aligned}
 \theta(0) = \theta(L) &= \frac{\pi}{2}, \\
 \phi(0) = \phi(L) &= 0,
 \end{aligned}
 \tag{6}$$

and the controlled end-rotation at both ends is incorporated into the boundary condition

$$\begin{aligned}
 \psi(0) &= 0, \\
 \psi(L) &= 2\pi M,
 \end{aligned}
 \tag{7}$$

for some non-zero integer M . The rod-energy is the sum of the strain energy (see (3)) and the potential energy as shown below

$$\mathcal{E}[\theta, \phi, \psi] = \int_0^L W(u_1, u_2, u_3) - F \sin \theta \cos \phi \, ds.
 \tag{8}$$

In (8), the strain components are related to the Euler angles as in (5) and F can be positive (stretching) or negative (compressive). An *equilibrium* is defined to be a solution of the Euler-Lagrange equations associated with (8) and a *stable equilibrium* is a local energy-minimizer in the space of admissible variations.

The unstressed state of the rod corresponds to the naturally straight state $\Theta_0(s) = (\theta_0(s), \phi_0(s), \psi_0(s))$ defined by

$$\begin{aligned}
 \theta_0(s) &= \frac{\pi}{2}, \quad 0 \leq s \leq L, \\
 \phi_0(s) &= 0, \quad 0 \leq s \leq L.
 \end{aligned}
 \tag{9}$$

The corresponding profile for ψ_0 will be determined from the Euler-Lagrange equations in the next section. We note that the first two strain components, u_1 and u_2 , identically vanish for the unstressed state. In the following sections, we consider two questions (i) for what choices of the matrix \mathbf{M} in (3) is the unstressed state an equilibrium state and (ii) for such choices of \mathbf{M} , when is the unstressed state a stable equilibrium and can we predict the onset of instabilities?

3 The Euler-Lagrange Equations

The energy density in (8) can be explicitly written in terms of the M_{ij} 's in (3) as shown below

$$e[\theta, \phi, \psi] = \frac{1}{2} \{ M_{11}u_1^2 + M_{22}u_2^2 + M_{33}u_3^2 + 2M_{12}u_1u_2 + 2M_{13}u_1u_3 + 2M_{23}u_2u_3 \} - F \sin \theta \cos \phi, \tag{10}$$

where the u_i 's are given in terms of the Euler angles in (5). However, it can be shown that the coefficient M_{12} must be zero [9]. To see this, consider the bending of the rod in a plane. The basis of the undeformed rod can always be chosen such that either u_1 or u_2 is zero and hence, the term $M_{12}u_1u_2$ must vanish in order to avoid dependence on the choice of the coordinate system. All equilibria associated with the rod-energy (8) are solutions of the following bulk Euler-Lagrange equations on the open interval $0 < s < L$ (see [12] for an exposition on the methods in the calculus of variations):

$$\frac{d}{ds} [\sin \theta [-M_{11}u_1 \cos \psi + M_{22}u_2 \sin \psi] + M_{33}u_3 \cos \theta + M_{13}[u_1 \cos \theta - u_3 \sin \theta \cos \psi] + M_{23}[u_2 \cos \theta + u_3 \sin \theta \sin \psi]] = F \sin \theta \sin \phi, \tag{11}$$

$$\frac{d}{ds} [M_{33}u_3 + M_{13}u_1 + M_{23}u_2] = u_1u_2(M_{11} - M_{22}) + (M_{13}u_2 - M_{23}u_1)u_3, \tag{12}$$

and

$$\begin{aligned} & \frac{d}{ds} [M_{11}u_1 \sin \psi + M_{22}u_2 \cos \psi + M_{13}u_3 \sin \psi + M_{23}u_3 \cos \psi] \\ & = \phi' \cos \theta [M_{22}u_2 \sin \psi - M_{11}u_1 \cos \psi] - M_{33}u_3 \phi' \sin \theta - F \cos \theta \cos \phi \\ & + M_{13} \phi' (-u_1 \sin \theta - u_3 \cos \theta \cos \psi) + M_{23} \phi' (-u_2 \sin \theta + u_3 \cos \theta \sin \psi). \end{aligned} \tag{13}$$

Equations (11)–(13) have been shown by O'Reilly [25] to be equivalent to the following decomposition of the angular momentum balance equations of a Cosserat rod

$$\begin{aligned} (11) & \Leftrightarrow (\mathbf{m}' + \mathbf{r}' \times \mathbf{n} = 0) \cdot \hat{\mathbf{z}}, \\ (12) & \Leftrightarrow (\mathbf{m}' + \mathbf{r}' \times \mathbf{n} = 0) \cdot \mathbf{d}_3, \\ (13) & \Leftrightarrow (\mathbf{m}' + \mathbf{r}' \times \mathbf{n} = 0) \cdot (\cos(\phi)\hat{\mathbf{y}} - \sin(\phi)\hat{\mathbf{x}}), \end{aligned} \tag{14}$$

where \mathbf{n} and \mathbf{m} are the respectively the resultant contact force and contact couple acting across a cross-section of the rod (see Antman [2] Chap. 8). For the straight axis solution, the correspondences (14) imply that \mathbf{m} is constant along the rod.

We now address the question—for what choices of $\{M_{11}, M_{22}, M_{33}, M_{13}, M_{23}\}$ is the unstressed state defined in (9) an equilibrium state? We denote the corresponding strain vector by $\mathbf{u}_0 = \{u_{10}, u_{20}, u_{30}\}$ with $u_{10} = u_{20} = 0$. We substitute $u_{10} = u_{20} = 0$ and $\theta_0(s) = \frac{\pi}{2}$ for $0 \leq s \leq L$ into (12) to find

$$M_{33}u_{30} = \text{constant}, \tag{15}$$

i.e., the unstressed state has constant twist density u_{30} . The equivalent constitutive relationship for the moment vector of this state is

$$\mathbf{m} = M_{13}u_{30}\mathbf{d}_1 + M_{23}u_{30}\mathbf{d}_2 + M_{33}u_{30}\mathbf{d}_3 \tag{16}$$

and hence (15) is equivalent to the moment conservation implied by the correspondences (14). Recalling that ϕ_0 is a constant from (9) and the controlled end-rotation boundary conditions (7), we necessarily have that

$$\psi'_0(s) = \frac{2\pi M}{L}$$

and hence,

$$\psi_0(s) = 2\pi M \frac{s}{L}, \quad 0 \leq s \leq L. \tag{17}$$

However, (11) requires that

$$M_{23}u_{30} \sin \psi_0 - M_{13}u_{30} \cos \psi_0 = \text{constant}$$

since $\phi_0 = 0$, and this is clearly incompatible with (17). We deduce that the off-diagonal elements

$$M_{13} = M_{23} = 0 \tag{18}$$

for the unstressed state $\Theta_0 = (\theta_0, \phi_0, \psi_0)$ to be an admissible equilibrium. Physically the constraints (18) imply that there is no coupling between bending and torsion.

Comment: We note that for the rod-energy (8) to have any straight equilibrium state characterized by $\theta(s) = C_1$ and $\phi(s) = C_2$, for constants C_1 and C_2 , we must have $M_{13} = M_{23} = 0$.

Equation (18) shows that it suffices to consider the *diagonal* rod-energy

$$\mathcal{E}_D[\theta, \phi, \psi] = \int_0^L \frac{1}{2} (Au_1^2 + Bu_2^2 + Cu_3^2) - F \sin \theta \cos \phi \, ds \tag{19}$$

whilst studying the unstressed state $\Theta_0 = (\theta_0, \phi_0, \psi_0)$ defined in (9) and (17) above. Here, $A, B, C > 0$ are material-dependent constants. If $A \neq B$, then the rod has an anisotropic cross-section whereas if $A = B$, then the rod cross-section is isotropic [21]. We refer to the $A \neq B$ case as the *anisotropic* case and to the $A = B$ case as the *isotropic* case in what follows. In the anisotropic case, we can always assume that

$$A \leq B \tag{20}$$

and this convention will be followed in the rest of the paper [16].

For completeness, the Euler-Lagrange equations associated with the diagonal rod-energy in (19) are given below,

$$\frac{d}{ds} [-Au_1 \sin \theta \cos \psi + Bu_2 \sin \theta \sin \psi + Cu_3 \cos \theta] = F \sin \theta \sin \phi, \tag{21}$$

$$C \frac{du_3}{ds} = u_1 u_2 (A - B), \tag{22}$$

$$\begin{aligned} &\frac{d}{ds} [Au_1 \sin \psi + Bu_2 \cos \psi] \\ &= \phi' \cos \theta [Bu_2 \sin \psi - Au_1 \cos \psi] - Cu_3 \phi' \sin \theta - F \cos \theta \cos \phi. \end{aligned} \tag{23}$$

We note that the corresponding equilibria do not have, in general, constant twist density u_3 for $A \neq B$.

4 Stability Analysis

4.1 The Anisotropic Case

In the previous section, we demonstrated that the unstressed state $\Theta_0 = (\theta_0, \phi_0, \psi_0)$, in (9) and (17), is an equilibrium state for the diagonal rod-energy in (19) for all choices of $A, B, C > 0$. In what follows, we take $M > 0$ and treat (6) and (7) collectively as a Dirichlet boundary-value problem for the Euler angles (θ, ϕ, ψ) . We first make the elementary observation that the unstressed state is not globally energy-minimizing for large, compressive forces $F < 0$ in (19).

Lemma 1 *For $F < 0$ and $|F|$ sufficiently large, the planar equilibrium $\Theta_0(s) = (\theta_0, \phi_0, \psi_0)$ for $s \in [0, L]$, is not a global minimizer for the rod-energy in (19), subject to the boundary conditions (6) and (7).*

Proof Recalling that $\theta_0 = \frac{\pi}{2}$, $\phi_0 = 0$ and $\psi_0(s) = 2\pi M \frac{s}{L}$, a direct computation shows that the rod-energy of the unstressed state is given by

$$\mathcal{E}_D[\theta_0, \phi_0, \psi_0] = 2\pi^2 C \frac{M^2}{L} + |F|L. \tag{24}$$

We construct a function $\theta^* : [0, L] \rightarrow \mathbb{R}$ subject to the clamped boundary conditions such that

$$\mathcal{E}_D[\theta^*, \phi_0, \psi_0] < \mathcal{E}_D[\theta_0, \phi_0, \psi_0] \tag{25}$$

for $|F|$ sufficiently large, thus proving the claim in Lemma 1. The triplet $\Theta^* = (\theta^*, \phi_0, \psi_0)$ need not be a solution of the Euler-Lagrange equations (21)–(23) but must satisfy the clamped boundary conditions in order to be an admissible construction. The function θ^* is defined to be

$$\theta^*(s) = \frac{\pi}{2} \left(1 - \frac{s(L-s)}{L^2} \right). \tag{26}$$

It is clear that $\theta^*(0) = \theta^*(L) = \frac{\pi}{2}$ as required from the clamped boundary condition.

The rod-energy of the triplet $\Theta^* = (\theta^*, \phi_0, \psi_0)$ is bounded from above by

$$\begin{aligned} \mathcal{E}_D[\theta^*, \phi_0, \psi_0] &\leq \frac{B}{2} \int_0^L \left(\frac{d\phi_0}{ds} \right)^2 \sin^2 \theta^*(s) + \left(\frac{d\theta^*}{ds} \right)^2 ds \\ &+ \int_0^L \frac{C}{2} \left(\frac{d\phi_0}{ds} \cos \theta^*(s) + \frac{d\psi_0}{ds} \right)^2 + |F| \sin \theta^*(s) ds \end{aligned} \tag{27}$$

since $A \leq B$ from (20). Since $\phi_0(s) = 0$ and $\psi_0(s) = 2\pi M \frac{s}{L}$ for $s \in [0, L]$, we have that

$$\frac{C}{2} \int_0^L \left(\frac{d\phi_0}{ds} \cos \theta^*(s) + \frac{d\psi_0}{ds} \right)^2 ds = 2\pi^2 M^2 \frac{C}{L}.$$

From the explicit form of θ^* in (26), we have

$$\int_0^L \left(\frac{d\theta^*}{ds} \right)^2 ds = \frac{\pi^2}{12L} \quad \text{and} \tag{28}$$

$$\int_0^L |F| \sin \theta^*(s) ds \leq |F| \frac{L}{2} + \frac{|F|L}{2} \sin \frac{13\pi}{32} \tag{29}$$

since $1 - \frac{s(L-s)}{L^2} \leq \frac{13}{16}$ for $s \in [\frac{L}{4}, \frac{3L}{4}]$. Substituting the above into (27), we obtain the following inequality

$$\mathcal{E}_D[\theta^*, \phi_0, \psi_0] \leq \frac{B\pi^2}{24L} + 2\pi^2 M^2 \frac{C}{L} + |F| \frac{L}{2} \left(1 + \sin \frac{13\pi}{32} \right). \tag{30}$$

We deduce that (25) will be satisfied whenever

$$\frac{B\pi^2}{24L} \leq |F|L \left(1 - \frac{1}{2} \left(1 + \sin \frac{13\pi}{32} \right) \right). \tag{31}$$

This inequality is satisfied for large $|F|$ since $(1 - \frac{1}{2}(1 + \sin \frac{13\pi}{32})) > 0$, hence proving the claim in Lemma 1. \square

The next step is to address the question of stability of the unstressed state and study the interplay between the elastic constants, twist, rod-length, and external force. To this end, we first recall a version of Weierstrass' theorem for one-dimensional problems [3, 19, 21]: consider the integral $I(y) = \int_a^b f(x, y(x), y'(x)) dx$ where $y'(x) = \frac{dy}{dx}$, subject to Dirichlet boundary conditions at the end-points $x = a$ and $x = b$. The Weierstrass theory guarantees that if f is strictly convex, i.e., its Hessian is positive definite, then an equilibrium solution $u \in C^2[a, b]$ is a local energy minimizer if the second variation $\delta^2 I[u] > 0$. We can explicitly compute the 3×3 Hessian matrix $\mathbf{T}(\Theta)$, of the energy density $e(\theta, \phi, \psi)$ in (19) as

$$\mathbf{T}(\Theta) = \begin{pmatrix} \frac{d^2 e}{d\theta'^2} & \frac{d^2 e}{d\theta' d\phi'} & \frac{d^2 e}{d\theta' d\psi'} \\ \frac{d^2 e}{d\theta' d\phi'} & \frac{d^2 e}{d\phi'^2} & \frac{d^2 e}{d\psi' d\phi'} \\ \frac{d^2 e}{d\theta' d\psi'} & \frac{d^2 e}{d\psi' d\phi'} & \frac{d^2 e}{d\psi'^2} \end{pmatrix}. \tag{32}$$

For the unstressed state defined in (9) and (17), this matrix is given by

$$\mathbf{T}(\Theta_0) = \begin{pmatrix} A \sin^2 \psi_0 + B \cos^2 \psi_0 & (B - A) \sin \psi_0 \cos \psi_0 & 0 \\ (B - A) \sin \psi_0 \cos \psi_0 & A \cos^2 \psi_0 + B \sin^2 \psi_0 & 0 \\ 0 & 0 & C \end{pmatrix} \tag{33}$$

and one can directly check that

$$h_i T(\Theta_0)_{ij} h_j = B(\cos \psi_0 h_1 + \sin \psi_0 h_2)^2 + A(\sin \psi_0 h_1 - \cos \psi_0 h_2)^2 + C h_3^2 > 0$$

for any non-zero vector $\mathbf{h} = (h_1, h_2, h_3) \in \mathbb{R}^3$. This necessarily implies that the eigenvalues of the matrix, $\mathbf{T}(\Theta_0)$, are positive. Hence, the energy density in (19) is strictly convex with respect to the first derivatives of the Euler angles in a neighborhood of the planar equilibrium Θ_0 and we can treat the positivity of the second variation as a sufficient criterion for local energy minimality [19].

For the sake of illustration, we repeat the same computation for the polar singularity $\theta = 0$, i.e., compute the Hessian $\mathbf{T}(\Theta)$ with $\theta = 0$. The corresponding matrix is given by

$$\mathbf{T}(\Theta) = \begin{pmatrix} A \sin^2 \psi + B \cos^2 \psi & 0 & 0 \\ 0 & C & C \\ 0 & C & C \end{pmatrix}$$

so that

$$h_i T(\Theta)_{ij} h_j = (A \sin^2 \psi + B \cos^2 \psi) h_1^2 + C(h_2 + h_3)^2$$

for any non-zero vector $\mathbf{h} = (h_1, h_2, h_3) \in \mathbb{R}^3$. If $\mathbf{h} \in \mathbb{R}^3$ is such that $h_1 = 0$ and $h_2 + h_3 = 0$, then the right-hand side of the above equality vanishes and hence, we lose strict convexity of the energy density with respect to the first derivatives at a polar singularity.

We non-dimensionalize the diagonal rod-energy in (19) by using the simple change of variable

$$\bar{s} = \frac{s}{L}, \quad 0 \leq \bar{s} \leq 1.$$

Then the re-normalized rod-energy is given by (up to a multiplicative constant)

$$\mathcal{E}_D[\theta, \phi, \psi] = \int_0^1 \frac{1}{2} (A u_1^2 + B u_2^2 + C u_3^2) - F L^2 \sin \theta \cos \phi \, d\bar{s}. \tag{34}$$

In what follows, we work with the re-normalized energy in (34) and drop the bars from the re-scaled variable for brevity. All subsequent results are to be understood in terms of the re-scaled arc-length variable above. The unstressed state is now defined by the triplet

$$\Theta_0(s) = (\theta_0, \phi_0, \psi_0) = \left(\frac{\pi}{2}, 0, 2\pi M s \right). \tag{35}$$

We consider in-plane perturbations (perturbations in the (xy) -plane) and out-of-plane perturbations (three-dimensional perturbations) separately. To this end, for a given $M > 0$, the general form of the in-plane perturbations of the unstressed state are given by $\Theta_\epsilon = (\theta_\epsilon, \phi_\epsilon, \psi_\epsilon)$ where

$$\begin{aligned} \theta_\epsilon(s) &= \frac{\pi}{2}, \quad 0 \leq s \leq L, \\ \phi_\epsilon(s) &= \epsilon \phi_1(s), \\ \psi_\epsilon(s) &= 2\pi M s + \epsilon \psi_1(s), \end{aligned} \tag{36}$$

and $\phi_1(0) = \phi_1(1) = 0$ and $\psi_1(0) = \psi_1(1) = 0$ by virtue of the Dirichlet conditions in (6) and (7). The out-of-plane perturbations are described by perturbations in the Euler angle θ and the general form $\Theta_\epsilon = (\theta_\epsilon, \phi_\epsilon, \psi_\epsilon)$ is defined to be

$$\begin{aligned} \theta_\epsilon(s) &= \frac{\pi}{2} + \epsilon \theta_1(s), \\ \phi_\epsilon(s) &= \epsilon \phi_1(s), \\ \psi_\epsilon(s) &= 2\pi M s + \epsilon \psi_1(s), \end{aligned} \tag{37}$$

where $\theta_1(0) = \theta_1(1) = 0$, $\phi_1(0) = \phi_1(1) = 0$ and $\psi_1(0) = \psi_1(1) = 0$ from the clamped boundary conditions (6) and the controlled end-rotation (7). We note that ϵ is sufficiently

small so that $\theta_\epsilon \in (0, \pi)$, i.e., θ_ϵ is bounded away from the polar singularities. From standard results in the calculus of variations [19, 26], the positivity of the second variation of the rod-energy

$$\frac{d^2}{d\epsilon^2} \mathcal{E}_D(\theta_\epsilon, \phi_\epsilon, \psi_\epsilon)|_{\epsilon=0} > 0 \tag{38}$$

is sufficient to guarantee the local energy minimality of the unstressed state. Similarly, if we can construct an explicit perturbation $\Theta_\epsilon = (\theta_\epsilon, \phi_\epsilon, \psi_\epsilon)$ for which

$$\frac{d^2}{d\epsilon^2} \mathcal{E}_D(\theta_\epsilon, \phi_\epsilon, \psi_\epsilon)|_{\epsilon=0} < 0, \tag{39}$$

then this is sufficient to deduce the instability of the unstressed state in the corresponding parameter regime.

Our first result concerns in-plane perturbations of the unstressed state where we recover the classical buckling result.

Proposition 1 *The unstressed state defined in (9) and (17) is stable with respect to in-plane perturbations of the form (36) if*

$$F + \frac{A\pi^2}{L^2} \geq 0 \tag{40}$$

where A is the smaller of the two bending stiffnesses $\{A, B\}$ from (20).

Proof Consider the perturbations defined in (36); these are the most general in-plane perturbations compatible with this boundary-value problem. We denote the corresponding perturbed strain/Darboux vector by $\mathbf{u}_\epsilon = (u_{1\epsilon}, u_{2\epsilon}, u_{3\epsilon})$ with strain components given by

$$\begin{aligned} u_{1\epsilon} &= -\epsilon\phi_1' \cos \psi_0, \\ u_{2\epsilon} &= \epsilon\phi_1' \sin \psi_0, \\ u_{3\epsilon} &= 2\pi M + \epsilon\psi_1'. \end{aligned} \tag{41}$$

We compute the corresponding second variation of the diagonal rod-energy in (19) to be

$$\begin{aligned} \frac{d^2}{d\epsilon^2} \mathcal{E}_D(\theta_\epsilon, \phi_\epsilon, \psi_\epsilon) \Big|_{\epsilon=0} &= \int_0^1 A \left(\frac{d\phi_1}{ds} \right)^2 \cos^2 \psi_0 + B \left(\frac{d\phi_1}{ds} \right)^2 \sin^2 \psi_0 + C \left(\frac{d\psi_1}{ds} \right)^2 \\ &\quad + FL^2 \phi_1^2 ds. \end{aligned} \tag{42}$$

The integral in (42) is bounded from below by

$$\frac{d^2}{d\epsilon^2} \mathcal{E}_D(\theta_\epsilon, \phi_\epsilon, \psi_\epsilon) \Big|_{\epsilon=0} \geq \int_0^1 A \left(\frac{d\phi_1}{ds} \right)^2 + C \left(\frac{d\psi_1}{ds} \right)^2 + FL^2 \phi_1^2 ds \tag{43}$$

since $A \leq B$ from (20). Recalling Wirtinger's inequality [28]

$$\int_0^1 \left(\frac{d\phi_1}{ds} \right)^2 ds \geq \pi^2 \int_0^1 \phi_1^2(s) ds,$$

we deduce that

$$\frac{d^2}{d\epsilon^2} \mathcal{E}_D(\theta_\epsilon, \phi_\epsilon, \psi_\epsilon) \Big|_{\epsilon=0} \geq \int_0^1 (A\pi^2 + FL^2) \phi_1^2 ds \tag{44}$$

and the second variation in (44) is clearly positive if $A\pi^2 + FL^2 > 0$ as stated in Proposition 1.

The borderline case $F = -A\frac{\pi^2}{L^2}$: for the general in-plane perturbations in (36), one can perform a Taylor series expansion for the energy of the triplet $(\theta_\epsilon, \phi_\epsilon, \psi_\epsilon)$ as shown below:

$$\begin{aligned} \mathcal{E}_D(\theta_\epsilon, \phi_\epsilon, \psi_\epsilon) &= \mathcal{E}_D(\theta_0, \phi_0, \psi_0) + \epsilon \frac{d\mathcal{E}_D(\theta_\epsilon, \phi_\epsilon, \psi_\epsilon)}{d\epsilon} \Big|_{\epsilon=0} + \frac{\epsilon^2}{2} \frac{d^2}{d\epsilon^2} \mathcal{E}_D(\theta_\epsilon, \phi_\epsilon, \psi_\epsilon) \Big|_{\epsilon=0} \\ &+ \frac{\epsilon^3}{6} \frac{d^3}{d\epsilon^3} \mathcal{E}_D(\theta_\epsilon, \phi_\epsilon, \psi_\epsilon) \Big|_{\epsilon=0} + \frac{\epsilon^4}{24} \frac{d^4}{d\epsilon^4} \mathcal{E}_D(\theta_\epsilon, \phi_\epsilon, \psi_\epsilon) \Big|_{\epsilon=0} + O(\epsilon^5). \end{aligned} \tag{45}$$

The first variation vanishes since $(\theta_0, \phi_0, \psi_0)$ is an equilibrium solution. The second variation is non-negative from (44). The third variation vanishes, as can be checked from the formula for the perturbed strain vector for in-plane perturbations from (41). Finally

$$\frac{d^4}{d\epsilon^4} \mathcal{E}_D(\theta_\epsilon, \phi_\epsilon, \psi_\epsilon) \Big|_{\epsilon=0} = -FL^2 \int_0^1 \phi_1^4(s) ds > 0$$

since $F = -A\frac{\pi^2}{L^2} < 0$. We conclude that the unstressed state is stable with respect to in-plane perturbations for $F \geq -A\frac{\pi^2}{L^2}$. □

We, next, address the question of stability with respect to out-of-plane perturbations.

Proposition 2 *The unstressed state, defined in (9) and (17), is a local minimizer of the rod-energy (19) (is nonlinearly stable) under the following conditions*

$$\begin{aligned} A &\geq 2MC, \\ F &> -A\frac{\pi^2}{L^2} + 2\pi^3 M\frac{C}{L^2} \quad \text{and} \\ F &> -A\frac{\pi^2}{L^2} + \frac{2\pi MC}{L^2}, \end{aligned} \tag{46}$$

where A is the smaller of the two bending stiffnesses $\{A, B\}$ from (20).

Proof Consider the perturbations defined in (37); these are the most general three-dimensional perturbations compatible with this boundary-value problem. We denote the corresponding perturbed Darboux vector by $\mathbf{u}_\epsilon = (u_{1\epsilon}, u_{2\epsilon}, u_{3\epsilon})$ with strain components given by

$$\begin{aligned} u_{1\epsilon} &= \epsilon(-\phi_1' \cos(2\pi Ms) + \theta_1' \sin(2\pi Ms)), \\ u_{2\epsilon} &= \epsilon(\phi_1' \sin(2\pi Ms) + \theta_1' \cos(2\pi Ms)), \\ u_{3\epsilon} &= 2\pi M + \epsilon\psi_1' - \epsilon^2\theta_1\phi_1'. \end{aligned} \tag{47}$$

The second variation of the rod-energy is computed to be

$$\begin{aligned} & \left. \frac{d^2}{d\epsilon^2} \mathcal{E}_D(\theta_\epsilon, \phi_\epsilon, \psi_\epsilon) \right|_{\epsilon=0} \\ &= \int_0^1 A(-\phi_1' \cos(2\pi Ms) + \theta_1' \sin(2\pi Ms))^2 ds \\ & \quad + \int_0^1 B(\phi_1' \sin(2\pi Ms) + \theta_1' \cos(2\pi Ms))^2 ds \\ & \quad + \int_0^1 C(\psi_1')^2 - 4\pi MC\phi_1'\theta_1 + FL^2(\theta_1^2 + \phi_1^2) ds, \end{aligned} \tag{48}$$

and is bounded from below by

$$\begin{aligned} \left. \frac{d^2}{d\epsilon^2} \mathcal{E}_D(\theta_\epsilon, \phi_\epsilon, \psi_\epsilon) \right|_{\epsilon=0} &\geq A \int_0^1 \left(\frac{d\theta_1}{ds}\right)^2 + \left(\frac{d\phi_1}{ds}\right)^2 ds + \int_0^1 C\left(\frac{d\psi_1}{ds}\right)^2 \\ &\quad - 4\pi MC\theta_1 \frac{d\phi_1}{ds} + FL^2(\theta_1^2 + \phi_1^2) ds. \end{aligned} \tag{49}$$

We partition the integral in (49) as shown below:

$$\begin{aligned} & \left. \frac{d^2}{d\epsilon^2} \mathcal{E}_D(\theta_\epsilon, \phi_\epsilon, \psi_\epsilon) \right|_{\epsilon=0} \\ &\geq 2\pi MC \int_0^1 \left(\frac{d\phi_1}{ds} - \theta_1\right)^2 ds + C \int_0^1 \left(\frac{d\psi_1}{ds}\right)^2 ds \\ &\quad + \int_0^1 (A - 2\pi MC) \left(\frac{d\phi_1}{ds}\right)^2 + FL^2\phi_1^2 ds \\ &\quad + \int_0^1 A\left(\frac{d\theta_1}{ds}\right)^2 + (FL^2 - 2\pi MC)\theta_1^2 ds. \end{aligned} \tag{50}$$

The integral $\int_0^1 2\pi MC\left(\frac{d\phi_1}{ds} - \theta_1\right)^2 ds + C \int_0^1 \left(\frac{d\psi_1}{ds}\right)^2 ds$ is non-negative since $M, C > 0$. If $A \geq 2\pi MC$, then we can apply Wirtinger's inequality to the ϕ -integral in (50) to get the following lower bound

$$\int_0^1 (A - 2\pi MC) \left(\frac{d\phi_1}{ds}\right)^2 + FL^2\phi_1^2 ds \geq \int_0^1 [(A - 2\pi MC)\pi^2 + FL^2]\phi_1^2 ds$$

and the integral in question is necessarily positive for a non-zero ϕ_1 if

$$[(A - 2\pi MC)\pi^2 + FL^2] > 0. \tag{51}$$

Similarly, we can obtain a lower bound for the θ -integral in (50),

$$\int_0^1 A\left(\frac{d\theta_1}{ds}\right)^2 + (FL^2 - 2\pi MC)\theta_1^2 ds \geq \int_0^1 [A\pi^2 + (FL^2 - 2\pi MC)]\theta_1^2 ds$$

and this integral is necessarily positive for a non-zero θ_1 if

$$[A\pi^2 + (FL^2 - 2\pi MC)] > 0. \tag{52}$$

We combine (51) and (52) to deduce that the unstressed state is a local minimizer of the rod-energy in (34) if

$$\min \left\{ A - 2\pi MC, A \frac{\pi^2}{L^2} + F - 2\pi M \frac{C}{L^2}, (A - 2\pi MC) \frac{\pi^2}{L^2} + F \right\} > 0. \tag{53}$$

The conclusion of Proposition 2 now follows. □

Proposition 3 *The unstressed state defined in (9) and (17) is unstable if*

$$F < -B \frac{\pi^2}{L^2}, \tag{54}$$

where B is the larger of the two bending stiffnesses from (20).

Proof We start with the integral formula for the second variation in (48) and make the elementary observation that

$$\begin{aligned} \frac{d^2}{d\epsilon^2} \mathcal{E}_D(\theta_\epsilon, \phi_\epsilon, \psi_\epsilon) \Big|_{\epsilon=0} &\leq \int_0^1 B \left[\left(\frac{d\phi_1}{ds} \right)^2 + \left(\frac{d\theta_1}{ds} \right)^2 \right] ds \\ &+ \int_0^1 C \left(\frac{d\psi_1}{ds} \right)^2 - 4\pi MC \theta_1 \frac{d\phi_1}{ds} + FL^2(\theta_1^2 + \phi_1^2) ds \end{aligned} \tag{55}$$

since $A \leq B$ from (20). We partition the integral in (55) as in (50):

$$\begin{aligned} \frac{d^2}{d\epsilon^2} \mathcal{E}_D(\theta_\epsilon, \phi_\epsilon, \psi_\epsilon) \Big|_{\epsilon=0} &\leq 2\pi MC \int_0^1 \left(\frac{d\phi_1}{ds} - \theta_1 \right)^2 ds + C \int_0^1 \left(\frac{d\psi_1}{ds} \right)^2 ds \\ &+ \int_0^1 (B - 2\pi MC) \left(\frac{d\phi_1}{ds} \right)^2 + FL^2 \phi_1^2 ds \\ &+ \int_0^1 B \left(\frac{d\theta_1}{ds} \right)^2 + (FL^2 - 2\pi MC) \theta_1^2 ds. \end{aligned} \tag{56}$$

We aim to construct a triplet $(\theta_1, \phi_1, \psi_1)$ for which $\frac{d^2}{d\epsilon^2} \mathcal{E}_D(\theta_\epsilon, \phi_\epsilon, \psi_\epsilon) \Big|_{\epsilon=0} < 0$ (recall the definition of $(\theta_\epsilon, \phi_\epsilon, \psi_\epsilon)$ from (37)). There are two possible options: (i) we can either construct $(\theta_1, \phi_1, \psi_1)$ so as to minimize the integrals $\int_0^1 (B - 2\pi MC) \left(\frac{d\phi_1}{ds} \right)^2 + FL^2 \phi_1^2 ds$ and $\int_0^1 B \left(\frac{d\theta_1}{ds} \right)^2 + (FL^2 - 2\pi MC) \theta_1^2 ds$ or (ii) construct $(\theta_1, \phi_1, \psi_1)$ so as to minimize the integral $\int_0^1 2\pi MC \left(\frac{d\phi_1}{ds} - \theta_1 \right)^2 + C \left(\frac{d\psi_1}{ds} \right)^2 ds$. In what follows, we construct perturbations $(\theta_1, \phi_1, \psi_1)$ which satisfy (i) and (ii) separately, compare the two different resulting instability criteria and choose the better one.

(i) We set

$$\begin{aligned} \phi_1(s) &= \sin(\pi s), \\ \theta_1(s) &= \sin(\pi s), \\ \psi_1(s) &= 0, \quad 0 \leq s \leq 1. \end{aligned} \tag{57}$$

We note that $\theta_1(0) = \theta_1(1) = 0$ and $\phi_1(0) = \phi_1(1) = 0$ as required from the clamped boundary condition (6). Substituting (57) into (55), we have that

$$\left. \frac{d^2}{d\epsilon^2} \mathcal{E}_D(\theta_\epsilon, \phi_\epsilon, \psi_\epsilon) \right|_{\epsilon=0} \leq 2(B\pi^2 + FL^2) \int_0^1 \sin^2(\pi s) ds, \tag{58}$$

where we have used the fact that $\int_0^1 (\frac{d \sin \pi s}{ds})^2 ds = \pi^2 \int_0^1 \sin^2(\pi s) ds$. It is clear that

$$\left. \frac{d^2}{d\epsilon^2} \mathcal{E}_D(\theta_\epsilon, \phi_\epsilon, \psi_\epsilon) \right|_{\epsilon=0} < 0$$

if

$$B\pi^2 + FL^2 < 0. \tag{59}$$

(ii) We construct ϕ_1 and θ_1 so that $\theta_1 = \frac{d\phi_1}{ds}$, subject to the clamped boundary condition (6). There are multiple choices but we set

$$\begin{aligned} \theta_1(s) &= \sin(2\pi s), \\ \phi_1(s) &= \frac{1}{2\pi} (1 - \cos(2\pi s)), \\ \psi_1(s) &= 0, \quad 0 \leq s \leq 1. \end{aligned} \tag{60}$$

We note that the functions θ_1 and ϕ_1 vanish at the end-points as required. Substituting (60) into the second variation formula (55), we find that

$$\left. \frac{d^2}{d\epsilon^2} \mathcal{E}_D(\theta_\epsilon, \phi_\epsilon, \psi_\epsilon) \right|_{\epsilon=0} \leq F \left(1 + \frac{3}{4\pi^2} \right) + 4B \frac{\pi^2}{L^2} + \frac{B}{L^2} - 4\pi M \frac{C}{L^2}$$

and the second variation is negative if

$$F < \frac{-4B \frac{\pi^2}{L^2} - \frac{B}{L^2} + 4\pi M \frac{C}{L^2}}{\left(1 + \frac{3}{4\pi^2} \right)}. \tag{61}$$

We compare formulae (59) and (61); (59) is the classical buckling result with bending stiffness B [26] and (61) depends on the controlled end-rotation characterized by M . One might expect (61) to contain more information when M is large. However, we want to compare instability results with the stability result in Proposition 2. Hence, we work with $B \geq A \geq 2\pi MC$. Then,

$$\frac{-4B \frac{\pi^2}{L^2} - \frac{B}{L^2} + 4\pi M \frac{C}{L^2}}{\left(1 + \frac{3}{4\pi^2} \right)} < -B \frac{\pi^2}{L^2}$$

since $2B\pi^2 \geq 4\pi MC$ and $1 + \frac{3}{4\pi^2} < 2$. We conclude that under the hypothesis $A \geq 2\pi MC$, (59) is a sharper result than (61) and we deduce that the unstressed state defined in (9) and (17) is unstable if

$$F < -B \frac{\pi^2}{L^2}. \tag{62} \quad \square$$

To summarize for the anisotropic case $A \neq B$ with three-dimensional perturbations, we have demonstrated stability of the unstressed state under the condition

$$\min \left\{ A - 2\pi MC, A \frac{\pi^2}{L^2} + F - 2\pi M \frac{C}{L^2}, (A - 2\pi MC) \frac{\pi^2}{L^2} + F \right\} > 0, \tag{62}$$

and instability for

$$F < -B \frac{\pi^2}{L^2}. \tag{63}$$

The two conditions are compatible since (62) guarantees stability for $A \geq 2\pi MC$ and

$$F > F_{c_1} = \max \left\{ -A \frac{\pi^2}{L^2} + 2\pi M \frac{C}{L^2}, -(A - 2\pi MC) \frac{\pi^2}{L^2} \right\}, \tag{64}$$

and (63) guarantees instability for $F < F_{c_2} = -B \frac{\pi^2}{L^2}$. The two results are consistent since $F_{c_2} < F_{c_1}$ from (20). The instability condition is analogous to the classical buckling result [26] and does not depend on the twist variable M . The stability result does depend on M and one can view F_{c_1} as the sum of the classical buckling critical force, $-A \frac{\pi^2}{L^2}$, with bending stiffness A , and a perturbation term, $\{2\pi M \frac{C}{L^2}, 2\pi MC \frac{\pi^2}{L^2}\}$, that depends on the torsional rigidity and the total twist. The stability result is affected by three-dimensional effects whereas the instability result can be captured by in-plane perturbations alone.

We note that the stability result in (2) may not be optimal. This is because the coupling term involving $\phi_1 \theta_1$, in (50), cannot be readily optimally estimated and is indeed the source of the mismatch between the stability and instability criteria. Should the condition $A \geq 2\pi MC$ be relaxed, then the stability criterion in Proposition 2 does not hold. However, Proposition 3 holds for all $A, M, C > 0$ and we have demonstrated that the unstressed state defined in (9) and (17) is unstable if

$$F < \min \left\{ -\frac{B\pi^2}{L^2}, \frac{-4B \frac{\pi^2}{L^2} - \frac{B}{L^2} + 4\pi M \frac{C}{L^2}}{(1 + \frac{3}{4\pi^2})} \right\}. \tag{65}$$

If $B \leq 2\pi MC$, then it is plausible that the second term in (65) yields a sharper upper bound for the critical force than the classical buckling result.

Comment: We can consider the more general situation of a rod aligned along a direction specified by the Euler angles $\theta = \theta_0 \in (0, \pi)$, $\phi = \phi_0$ with the external force being applied along the same direction. The corresponding energy is

$$\begin{aligned} E[\theta, \phi, \psi] = & \int_0^1 \frac{A}{2} \left(\left(\frac{d\theta}{ds} \right)^2 + \left(\frac{d\phi}{ds} \right)^2 \sin^2 \theta \right) + \frac{C}{2} \left(\frac{d\phi}{ds} \cos \theta + \frac{d\psi}{ds} \right)^2 ds \\ & - \int_0^1 FL^2 (\sin \theta \cos \phi \sin \theta_0 \cos \phi_0 + \sin \theta \sin \phi \sin \theta_0 \sin \phi_0 + \cos \theta_0 \cos \theta) ds \end{aligned} \tag{66}$$

subject to the boundary conditions $\theta(0) = \theta(1) = \theta_0$, $\phi(0) = \phi(1) = \phi_0$, $\psi(0) = 0$, $\psi(1) = 2\pi M$. One can check that $\theta(s) = \theta_0$, $\phi(s) = \phi_0$ for $0 \leq s \leq L$ is an equilibrium solution. Further, the stability results in Propositions 2 and 3 can be generalized to this physical problem too. In particular, the corresponding stability criterion is that the unstressed naturally

straight state defined by, $\Theta = (\theta_0, \phi_0, 2\pi Ms)$, is a local minimizer of the energy (66) for forces

$$F > \max \left\{ -[(A - 2\pi MC) + C \cot^2 \theta_0] \frac{\pi^2}{L^2}, -A \frac{\pi^2}{L^2} + \frac{2\pi MC}{L^2} \right\} \tag{67}$$

under the constraint

$$(A - 2\pi MC) \sin^2 \theta_0 + C \cos^2 \theta_0 > 0.$$

We recover the stability criterion in Proposition 2 when $\theta_0 = \frac{\pi}{2}$.

4.2 The Isotropic Case

For an elastic rod with isotropic cross-section, the bending rigidities are equal, i.e., $A = B$ in (19) and the rod-energy reduces to the simpler form

$$\begin{aligned} \mathcal{E}_I[\theta, \phi, \psi] = & \int_0^1 \frac{A}{2} \left[\left(\frac{d\phi}{ds} \right)^2 \sin^2 \theta(s) + \left(\frac{d\theta}{ds} \right)^2 \right] + \frac{C}{2} \left(\frac{d\phi}{ds} \cos \theta(s) + \frac{d\psi}{ds} \right)^2 \\ & - FL^2 \sin \theta(s) \cos \phi(s) ds \end{aligned} \tag{68}$$

subject to the boundary conditions $\theta(0) = \theta(1) = \frac{\pi}{2}$, $\phi(0) = \phi(1) = 0$ and $\psi(0) = 0$, $\psi(1) = 2\pi M$. The corresponding equilibria are solutions of the associated Euler-Lagrange equations:

$$\begin{aligned} \phi' \cos \theta + \psi' &= u_3 = \text{constant}, \\ \frac{d}{ds} [A\phi' \sin^2 \theta + Cu_3 \cos \theta] &= FL^2 \sin \theta \sin \phi, \\ A \frac{d^2 \theta}{ds^2} &= A \sin \theta \cos \theta \left(\frac{d\phi}{ds} \right)^2 - FL^2 \cos \theta \cos \phi - Cu_3 \sin \theta \frac{d\phi}{ds}. \end{aligned} \tag{69}$$

We note that unlike the anisotropic case, all equilibria have constant twist density u_3 .

The unstressed state defined in (9) and (17) is an equilibrium state for the isotropic rod-energy in (68), i.e., is a solution of the Euler-Lagrange equations in (69). We define F_c to be the critical force such that the unstressed state is a stable equilibrium for $F \geq F_c$ and is unstable for $F < F_c$. We can obtain explicit upper and lower bounds for F_c from Propositions 2 and 3, by simply setting $A = B$. This yields the following estimates for F_c under the hypothesis $A \geq 2\pi MC$ as shown below:

$$-A \frac{\pi^2}{L^2} < F_c < \max \left\{ -A \frac{\pi^2}{L^2} + 2\pi M \frac{C}{L^2}, -(A - 2\pi MC) \frac{\pi^2}{L^2} \right\}. \tag{70}$$

In the special case of in-plane perturbations, we recall Proposition 1 and deduce that the unstressed state is stable with respect to in-plane perturbations if $F \geq -A \frac{\pi^2}{L^2}$ and unstable otherwise.

5 Conclusion

We study the stability of a naturally straight Kirchhoff rod, aligned along the x -axis, subject to clamped boundary conditions, controlled end-rotation and terminal loads. We start

with a general rod-energy in (8) and show that the diagonal rod-energy in (19) is the most general rod-energy compatible with any planar equilibrium state. We approach the stability analysis from a variational perspective and study the second variation of the diagonal rod-energy, including three-dimensional effects into the problem formulation. The second variation methodology is a more powerful technique than static bifurcation methods and yields rigorous results on the stability spectrum.

By means of a direct computation, we demonstrate that the energy density in (19) is strictly convex with respect to the first derivatives of the Euler angles provided we do not encounter the polar singularities, $\theta = 0$ or $\theta = \pi$. The strict convexity is a pre-requisite for the application of Weierstrass theory for scalar variational problems [3, 19] to our model problem. We use the positivity of the second variation of the rod-energy as a sufficient criterion for the local energy minimality of the unstressed naturally straight state. From Proposition 2, the naturally straight state is stable for external forces

$$F > \max \left\{ -A \frac{\pi^2}{L^2} + 2\pi M \frac{C}{L^2}, -(A - 2\pi MC) \frac{\pi^2}{L^2} \right\}$$

provided $A \geq 2\pi MC$. More precisely, we show that in the parameter regime (64), there exists an ϵ -neighborhood, \mathcal{A}_ϵ , of the unstressed state Θ_0 in (9) and (17) for some suitably defined $\epsilon > 0$, i.e.,

$$\mathcal{A}_\epsilon = \left\{ (\theta, \phi, \psi) \in \mathbb{R}^3; \max \left\{ \left| \theta(s) - \frac{\pi}{2} \right|, |\phi(s)|, |\psi(s) - 2\pi Ms| \right\} \leq \epsilon \right\} \quad (71)$$

such that

$$\mathcal{E}_D(\Theta_0) < \mathcal{E}_D(\theta, \phi, \psi) \quad \forall (\theta, \phi, \psi) \in \mathcal{A}_\epsilon. \quad (72)$$

This conclusion follows directly from the positivity of the second variation of the rod-energy and is the definition of a *strict local energy minimum* [3, 19]. This stability result is accompanied by an instability result in Proposition 3 where we demonstrate instability of the naturally straight state for $F < -B \frac{\pi^2}{L^2}$. Therefore, if F_c denotes the critical force such that the naturally straight state is stable for $F \geq F_c$ and unstable for $F < F_c$, then

$$-B \frac{\pi^2}{L^2} < F_c < \max \left\{ -A \frac{\pi^2}{L^2} + 2\pi M \frac{C}{L^2}, -(A - 2\pi MC) \frac{\pi^2}{L^2} \right\}, \quad (73)$$

where $A \leq B$. The two-dimensional in-plane analysis is simpler whereby we recover the classical buckling result in Proposition 1.

The results in this paper are an example of how variational methods can be applied to stability problems in the theory of Kirchhoff rods, yielding rigorous results in a three-dimensional framework. Our methods highlight the mathematical analogies between the anisotropic case ($A \neq B$) and the isotropic case ($A = B$), see Propositions 2 and 3 where we effectively reduce the anisotropic problem to the isotropic one. We use integral inequalities to obtain bounds for the second variation and this is an approach that can be further exploited. We hope to generalize these techniques to a wide class of problems in elasticity with special emphasis on the structure and stability of buckled modes, subject to topological constraints, e.g., fixed linking number [13] in future work.

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