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Helices through 3 or 4 points?

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Abstract. How many points in space are needed to define a circular helix? We show here that given 3 distinct points in space there exist continuous families of helices passing through these points. Given 4 generic distinct points there is no helix. However, a discrete family of helices through 3 points can be specified if an additional property of the helix is prescribed. In particular, the case where the helical radius is specified is studied in detail.

1 Introduction

A classical problem of Cartesian geometry is to obtain a curve or a surface containing a given set of points. The first part of the problem is to identify the minimum number of points leading to either a unique solution, or a discrete family of solutions. For instance, we learn in school that, with the usual restrictions to avoid singular configurations, 2 points are needed for a line, and 3 points are needed to define a plane. Other classical results are that, generically, one needs 3 points to define a unique circle and 4 points for a sphere. Similarly, but not commonly known, there exists up to 6 cylinders through 5 points in space [1]. The case of 6 cylinders can be obtained by locating the points at the vertices of a bipyramid obtained by identifying the faces of two regular tetrahedra. The second part of the problem is to obtain a complete characterization of the object of interest and to provide a simple way of deriving its equation. There exist elegant formulas for planes, lines, circles, and spheres. The case of cylinders requires however the solution of algebraic equations but it still leads to a simple algorithmic procedure.

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Helices with their constant curvature and torsion are arguably the simplest non-planar curves in the three-dimensional Euclidean space. Clearly, the elementary problem of finding the number of points necessary to define a helix begs for a solution. However, since this problem can be answered with classical analytical geometry, it does not seem worth the attention or time of the modern professional mathematician and, indeed, it does not seem to have been addressed. Helices are also ubiquitous in many scientific fields [2] such as biochemistry [3], elementary particle physics [4], computer sciences (finite element codes [5] or visualization [6]), and, to date, there exist many sophisticated techniques to find the best fitting helix through a cloud of points [7, 8, 9]. Nevertheless, the problem of finding the number of points necessary to define a single helix has not been addressed in these fields either. The main reason may be that there is no simple solution to the problem. Indeed, we show here that there exists a one-parameter family of helices passing through 3 points but that no helix passes through 4 generic points in space. However, these results can be used constructively in three ways. First, we show that there exists a countable set of helices passing through 3 points lying on a cylinder defined by 4 points. Second, we show that there exists a countable set of helices through three points of a given (large enough) radius. Third, for 2 points, there exists a discrete family of helices containing the 2 points and with a prescribed tangent, normal and binormal vectors at one of the points. This last property has been exploited by the authors to identify sequences of helical segments through n points in a related paper [10] and will not be discussed here.

2 Definitions

2.1 Geometry of helices

Before proceeding with the construction of helices, we recall some basic property of helical geometry. First, we consider a curve $\mathbf{r}(s) = (x(s), y(s), z(s))$, of class C^3, parametrized by its arc length s in a fixed reference frame $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$. From the curve, we can define the Frenet basis, that is a local orthonormal basis on $\mathbf{r}$ defined by the tangent vector $\mathbf{t} = \mathbf{r}'$ as the arc-length derivative of $\mathbf{r}$, the normal $\mathbf{n} = \mathbf{t}'/|\mathbf{t}'|$ and the binormal $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ (the prime $(\ )'$ denotes differentiation with respect to s). The changes in the orientation of this frame along s are specified by the Frenet equations in terms of two local quantities the curvature $\kappa(s)$ and torsion $\tau(s)$:

$$\mathbf{r}' = \mathbf{t}, \tag{1}$$
$$\mathbf{t}' = \kappa\mathbf{n}, \tag{2}$$
$$\mathbf{n}' = \tau\mathbf{b} - \kappa\mathbf{t}, \tag{3}$$
$$\mathbf{b}' = -\tau\mathbf{n}. \tag{4}$$

A *circular helix* or simply a *helix* is defined as a curve with constant curvature and torsion. To relate the curvature and torsion to the usual radius and pitch of a helix, without loss of generality, we can study a helix along the z-axis

$$\mathbf{r} = (R\cos(\delta s), R\sin(\delta s), P\delta s)\,, \quad \text{where} \quad \delta = \frac{1}{\sqrt{P^2 + R^2}}. \tag{5}$$

The choice $P > 0$ (resp. $P < 0$) defines a right-handed helix (resp. left-handed) as shown in Fig. 1. The *height* or *pitch* (along the z-axis) per turn of the helix is $p = 2\pi|P|$, the radius R, and the length of the curve per turn is $2\pi/\delta$. We

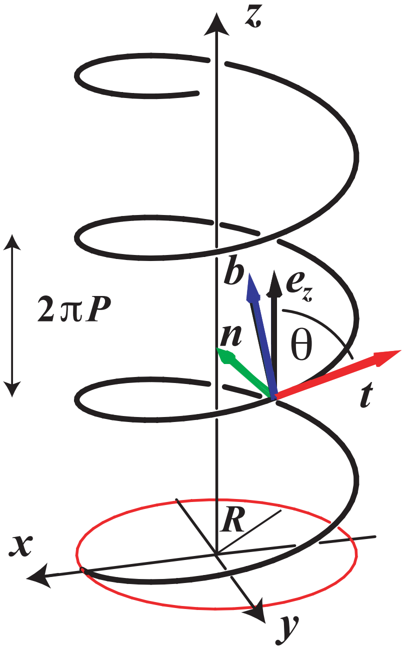

Figure 1. A helix characterized by a radius R, and a pitch $2\pi P$. The local Frenet frame is shown at one particular point. The angle θ is the angle between the tangent vector $\mathbf{t}$ and the helix axis $\mathbf{e}_z$.

can now build the Frenet triad for the helix. The normal vector $\mathbf{n}$ is obtained by further differentiating and normalizing the tangent vector and the binormal vector $\mathbf{b}$ is simply constructed by taking the cross product $\mathbf{b} = \mathbf{t} \times \mathbf{n}$.

$$\mathbf{t} = (-R\delta \sin \delta s, R\delta \cos \delta s, P\delta)\,, \tag{6}$$
$$\mathbf{n} = (-\cos \delta s, -\sin \delta s, 0)\,, \tag{7}$$
$$\mathbf{b} = (P\delta \sin \delta s, -P\delta \cos \delta s, R\delta)\,. \tag{8}$$

The curvature, κ and torsion, τ, are obtained by considering the norm of $\mathbf{t}'$ and $\mathbf{n}'$ and are found to be

$$\kappa = R\delta^2 = \frac{R}{P^2 + R^2}, \quad \tau = P\delta^2 = \frac{P}{P^2 + R^2}, \tag{9}$$

which implies $\delta^2 = \kappa^2 + \tau^2$ and

$$R = \kappa/\delta^2 = \frac{\kappa}{\kappa^2 + \tau^2}, \quad P = \tau/\delta^2 = \frac{\tau}{\kappa^2 + \tau^2}, \tag{10}$$

The *helix angle* θ is the angle between the axis and the tangent vector defined in the interval $[0, \pi]$ by

$$\theta = \arccos(\mathbf{t} \cdot \mathbf{e}_z) = \arccos(P\delta) = \arccos(\tau/\delta). \tag{11}$$

Similarly, the *pitch angle* $\widehat{\theta}$ is the angle between the tangent and the plane normal to the axis, that is $\widehat{\theta} = \pi/2 - \theta$. The sign of the pitch angle (or, equivalently, the sign of the torsion) defines the handedness of the helix (right-handed for positive pitch angles, left-handed, otherwise). The pitch angle is related to the pitch and radius by

$$\cos \widehat{\theta} = R\delta, \quad \sin \widehat{\theta} = P\delta. \tag{12}$$

Finally, the cylindrical coordinates of points on the helix will be of some use. Given a reference point A and a point B on the helix separated by an arc length s_B, the cylindrical coordinates (R, φ, z) of the point B with respect to A are given by

$$s_A = 0,\ Z_A = 0,\ \varphi_A = 0, \tag{13}$$
$$Z_B = Z(s_B) = s_B \cos \theta,\ \varphi_B = \widehat{\varphi}(s_B) \mod 2\pi \tag{14}$$

where $\widehat{\varphi}(s_B) = \frac{s_B}{R} \sin \theta$. Therefore, the equation of a helix in cylindrical coordinates is simply a straight line with

$$Z = \frac{R}{\tan \theta} \widehat{\varphi} = P\widehat{\varphi}, \tag{15}$$

that is a straight line in the $(\widehat{\varphi}, Z)$-plane with slope P.

Note that given 2 points on a cylinder, there exists a countable set of helices passing through these points. The best way to visualize a helix on a cylinder is to consider the $(\widehat{\varphi}, Z)$-plane which corresponds to cutting the cylinder along the line parallel to its axis and passing through the point A, unrolling it, and extending periodically the strip along the $\widehat{\varphi}$ axis. On that plane, a helix is a straight line through the points $(\varphi_A, Z_A) = (0, 0)$ and (φ_B, Z_B) with a slope $P = Z_B/\varphi_B$. However, since the angle φ is given up to an arbitrary multiple of 2π, there exists a countable set of straight lines through the points (φ_A, Z_A) and $(\varphi_B + 2k\pi, Z_B)$, k an integer, with slopes $P = Z_B/(\varphi_B + 2k\pi)$ (See Fig. 2). This construction provides a simple test to verify whether there exists a helix

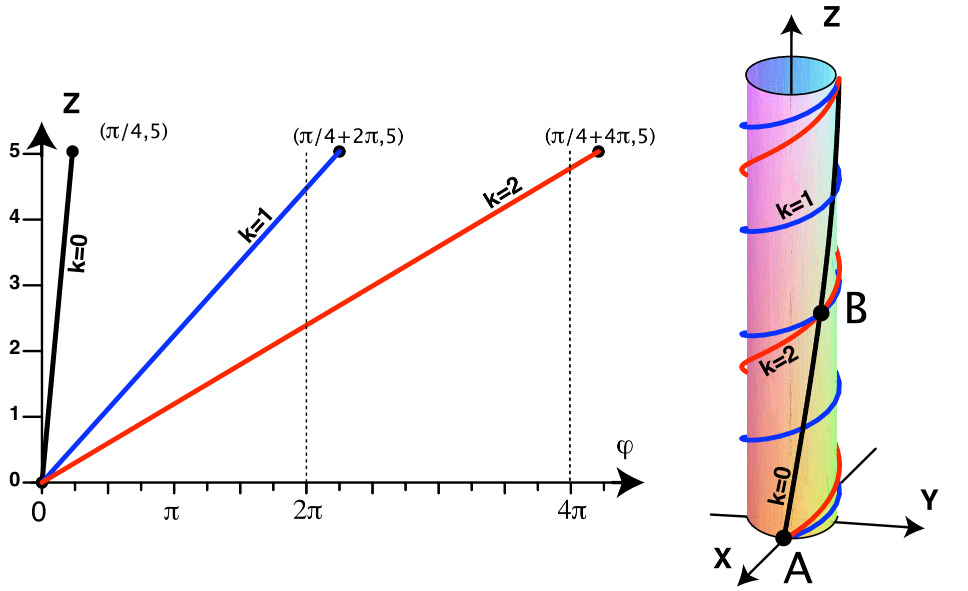

Figure 2. Two points on a cylinder define a discrete family of helices. Three helices passing by the points $A = (1, 0, 0)$ and $B = (1/\sqrt{2}, 1/\sqrt{2}, 5)$ and corresponding to the choices $k = 0, 1, 2$. In the (φ, Z)-plane, each helix corresponds to a straight line through the points $(0, 0)$ and $(\pi/4 + 2k\pi, 5)$.

through a set of points on a cylinder.

Lemma 1. A set of $n > 2$ points on a cylinder with cylindrical coordinates (R, φ_i, Z_i), $i = 0, \ldots, n-1$ lie on a helix if there exist $(n-1)$ integers $k_1, \ldots, k_{n-1}$ such that the points $(\varphi_i + 2\pi k_i, Z_i)$, $i = 0, \ldots, n-1$ with $k_0 = 0$ are colinear in the $(\widehat{\varphi}, Z)$-plane.

Accordingly, a helix passes through n points on a cylinder if $(n-2)$ conditions are satisfied (up to an integer lattice). Whenever two, three, or four points are given and we look for a helix through these points, we define the *principal helix* as the helix corresponding to $k_i = 0$ for all i, that is, the helix with the smaller angular increase between the first and last points. In the case of a cylinder and two points, the principal helix corresponds to the choice $k_1 = 0$.

3 Helices through 4 points

We first consider the case of 4 points in space. Our first result is negative.

Theorem 2. Generically, there is no helix passing through 4 or more points in $\mathbb{R}^3$.

Proof. We consider 4 points $\mathbf{P}_i \in \mathbb{R}^3,\ i = 0, 1, 2, 3$. We identify the Cartesian coordinates of a point $P = (a, b, c)$ with the components of the vector $\mathbf{P} = (a, b, c)$ from the origin to the point P in the canonical basis. We choose, without loss of generality, $\mathbf{P}_0 = (0, 0, 0)$ and define the plane

$$\Pi: \ \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0, \tag{16}$$

as the plane passing through the origin and perpendicular to the unit vector $\boldsymbol{\alpha} = (\sin\xi \cos\eta, \sin\xi \sin\eta, \cos\xi)$ where $\xi \in [0, \pi]$ and $\eta \in [0, \pi]$.

In order to find a helix through these 4 points, we consider their projections on the plane Π and require that all 4 points lie on a circle in the plane. This provide a first condition on the coefficient $\boldsymbol{\alpha}$. The projections of the 4 points are

$$\mathbf{Q}_i = \mathbf{P}_i - Z_i \boldsymbol{\alpha}, \quad Z_i = \boldsymbol{\alpha} \cdot \mathbf{P}_i, \qquad i = 0, \ldots, 3, \tag{17}$$

where Z_i is the Z-cylindrical coordinates of $\mathbf{P}_i$ with respect to the plane Π. The condition that the 4 points $\{\mathbf{Q}_0, \mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3\}$ lie on a circle is provided by Ptolemy's theorem which states that a necessary and sufficient condition for a convex quadrilateral to be inscribed in a circle is that the sum of the products of the two pairs of opposite sides equal the product of the diagonals [11]. In our case, we consider the quadrilateral defined by the vertices $\{\mathbf{Q}_0, \mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3\}$ on Π and we let d_{ij} be the distance between vertices $\mathbf{Q}_i$ and $\mathbf{Q_j}$. Ptolemy's theorem states that the condition for the points to lie on a circle is

$$C_1 \stackrel{\text{def}}{=} d_{01} d_{23} - d_{03} d_{12} - d_{02} d_{13} = 0. \tag{18}$$

This condition provides a relationship on the angles ξ and η, that is, it defines a one-parameter family of cylinders containing the four points. According to Lemma 1, on any given cylinder, a helix passes through the 4 points $\mathbf{P_i},\ i =$

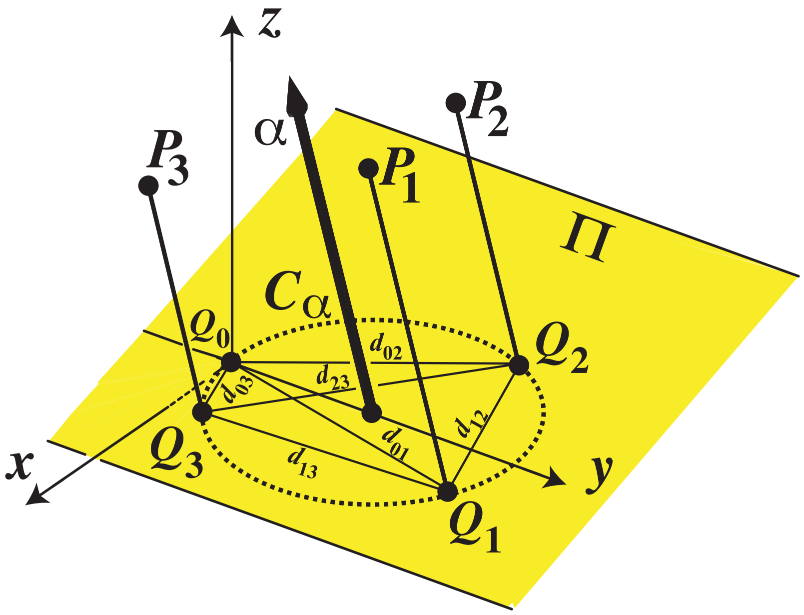

Figure 3. The projection of 3 points $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$ on the particular plane Π chosen such that the projected points $\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3$ lie on a circle passing through $\mathbf{P}_0 = \mathbf{Q}_0$. The four points on the plane Π define a quadrilateral.

$0,\ldots,3$ if two conditions are satisfied, which, together with condition C_1 provide three independent conditions for the two free parameters ξ and η. We conclude that, generically, there is no helix through 4 points in space. QED

Unfortunately, there is no simple condition such as Ptolemy's theorem to guarantee that 4 points in space lie on a helix. Such a result would be highly desirable since the fitting of helices through points in space is an important problem in many scientific fields. To test whether 4 points lie on a helix, we can first construct a helix through 3 points lying on cylinder containing 4 points, then check whether the fourth points is also on the helix. This is based on the following result.

Proposition 3. Given 4 points in $\mathbb{R}^3$, there exists a countable set (possibly empty) of cylinders containing the 4 points and a helix passing through 3 of the points.

Proof. The proof follows directly from the previous construction. The condition C_1 provides a one-parameter family of cylinder containing the 4 points $\mathbf{P_i},\ i=0,\ldots,3$. Now, consider, without loss of generality, the three points $\mathbf{P_i},\ i=0,\ldots,2$. The further condition that they lie on a helix provides for each pair of integers (k_1,k_2) a unique relationship on the free parameters ξ,η in the one-parameter family. However, the existence of solutions for these transcendental equations cannot be guaranteed and the solution set may be empty. QED

4 Helices through 3 points

Since 4 points provide too many conditions to define a helix, the next logical step is to consider 3 points. However, 3 points do not provide enough conditions to define a discrete family of helices.

Theorem 4. There exists an uncountable set of helices through 3 non-colinear points in $\mathbb{R}^3$.

Proof. We follow the construction in the previous section and consider the projection of three points in a plane through the origin. For almost all ξ and η the three points on the plane Π define a circle and a cylinder of axis $\boldsymbol{\alpha}$. The colinearity condition on the variable (φ,Z) that the three points lie on a helix provide for each pair of integer (k_1,k_2) one relationship between the free parameters ξ and η. We conclude that there exists a discrete set of one-parameter families of helices through three points in space. QED

This result cannot be used for an explicit construction of helices through three points. However, if we further restrict the helices passing through the points by specifying one of the helical parameters, we can obtain some interesting results. Many different properties could be specified such as torsion, curvature, pitch, or radius. Here we consider the case of fixed radius.

4.1 Helices through 3 points with given radius

Proposition 5. Given a strictly positive real number R and 3 points in $\mathbf{R^3}$, there exists a discrete set (possibly empty) of helices with radius R and passing through the points.

Proof. Following the previous result, we consider the projection of three points in a plane through the origin. For almost all ξ and η the three points on the plane Π define a circle and a cylinder of axis $\boldsymbol{\alpha}$ with two free parameters. A countable set of solutions for these parameters is obtained as the set of solution of the system formed by the colinearity condition and the condition that the circle on the plane Π has radius R. The fact that the set of solution is possibly empty for a given radius should be clear when one consider the circle on the plane Π passing through the projection $\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3$ of the 3 points $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$. By contradiction, unless the three points $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$ lie on a line, the circle will have a radius $R > R_{\min} > 0$. Therefore, there exist radii $R < R_{\min}$ for which no helix exists. QED

On the contrary, choosing a radius sufficiently large always guarantees the existence of helices.

Proposition 6. For 3 non-colinear points in $\mathbf{R^3}$, there exists a real number $R_{\max}$ such that for all $R > R_{\max}$, there exists a discrete set of helices with radius R and passing through the points.

Proof. The strategy of the proof is as follows. Without loss of generality, we choose the three points to be $\mathbf{P}_0 = (0,0,0)$, $\mathbf{P}_1 = (1,0,0)$, and $\mathbf{P}_2 = (a,b,0)$. We consider the projection $\mathbf{Q}_0 = \mathbf{P}_0$, $\mathbf{Q}_1$ and $\mathbf{Q}_2$ of these points in a plane close to the plane perpendicular to the $z = 0$ plane by choosing the vector $\boldsymbol{\alpha}$ to be $\boldsymbol{\alpha} = (\sin\xi\cos\eta, \sin\xi\sin\eta, \cos\xi)$ with $\xi = \frac{\pi}{2} - \epsilon$, $\eta = \frac{\pi}{2} - A\epsilon$, and $\epsilon > 0$ sufficiently small. The vector $\boldsymbol{\alpha}$ is then $\boldsymbol{\alpha} = (A\epsilon, 1, \epsilon) + O(\epsilon^2)$. As $\epsilon \to 0$ the radius of the circle passing through $\{\mathbf{Q}_0, \mathbf{Q}_1, \mathbf{Q}_2\}$ becomes unbounded. We then show that there exists arbitrary small values of ϵ such that the points $\{\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2\}$ lie on a helix.

Explicitly, the projected points are

$$\mathbf{Q}_1 = (1, -A\epsilon, 0) + O(\epsilon^2), \tag{19}$$
$$\mathbf{Q}_2 = (a - bA\epsilon, -Aa\epsilon, -b\epsilon) + O(\epsilon^2). \tag{20}$$

The radius R of the circle through $\{\mathbf{Q}_0, \mathbf{Q}_1, \mathbf{Q}_2\}$ is

$$R = \frac{|a(a-1)|}{2|b|\epsilon} + O(1), \tag{21}$$

which establishes the fact that R becomes unbounded as $\epsilon \to 0$. It remains to show that for R sufficiently large there exists a helix. The cylindrical coordinates (Z_1, φ_1) of $\mathbf{P}_1$ and (Z_2, φ_2) of $\mathbf{P}_2$ with respect to the projection plane are

$$Z_1 = A\epsilon + O(\epsilon^2), \qquad \varphi_1 = \arg\left(1 + i\frac{2b\epsilon}{a(a-1)}\right) + O(\epsilon^2), \tag{22}$$

$$Z_2 = b + Aa\epsilon + O(\epsilon^2), \quad \varphi_2 = \arg\left(1 - i\frac{2b\epsilon}{1-a}\right) + O(\epsilon^2), \tag{23}$$

with

$$\arg(z) = \begin{cases} \mathrm{Arg}(z) & \text{if } \Im(z) \geq 0 \\ 2\pi + \mathrm{Arg}(z) & \text{otherwise,} \end{cases} \tag{24}$$

and $\mathrm{Arg}(z)$ is the principal argument of the complex number z and $\arg(z) \in [0; 2\pi)$. Without loss of generality, we look at the case $a < 0$ and $b > 0$ where $\varphi_1 = \frac{2b\epsilon}{a(a-1)}$ and $\varphi_2 = 2\pi - \frac{2b\epsilon}{1-a}$. The condition for the three points $\mathbf{P}_0$, $\mathbf{P}_1$, and $\mathbf{P}_2$ to lie on a helix is then

$$-2bk_1\pi + 2\left(A\pi(1 - ak_1 + k_2) - \frac{b^2}{a(a-1)}\right)\epsilon + O(\epsilon^2) = 0. \tag{25}$$

This condition is valid for ϵ sufficiently small and can be satisfied by choosing $k_1 = 0$, $k_2 \neq 0$ and $A = b^2/(a(a-1)(k_2+1)\pi)$. Therefore, we conclude that for all ϵ sufficiently small, there exists a helix passing through three non-colinear points. The smallest corresponding radius $R_{\max}$ corresponds to the largest possible value of ϵ guaranteeing the convergence of all the series involved in the computation. *QED*

4.2 A method for constructing helices through 3 points with a given radius

We choose the reference frame such that one of the points, say $\mathbf{P}_0$, is at the origin. The two other points are labeled $\mathbf{P}_1$ and $\mathbf{P}_2$. Following the proofs, the main idea of the construction is to consider the projection, along the direction $\boldsymbol{\alpha}$, of these three points onto a plane Π comprising the origin. Once the points $\mathbf{Q}_0 = \mathbf{P}_0 = (0,0,0)$, $\mathbf{Q}_1$, and $\mathbf{Q}_2$ are found (See Fig. 4), we find the circle through the three projected points. The circle and the unit vector $\boldsymbol{\alpha}$ define a cylinder. We obtain the cylindrical coordinates of the original points with respect to this cylinder and enforce the colinearity condition on the cylinder, which together with the condition on the radius, provide a system of two equations for the two unknown angles defining the unit vector perpendicular to the plane.

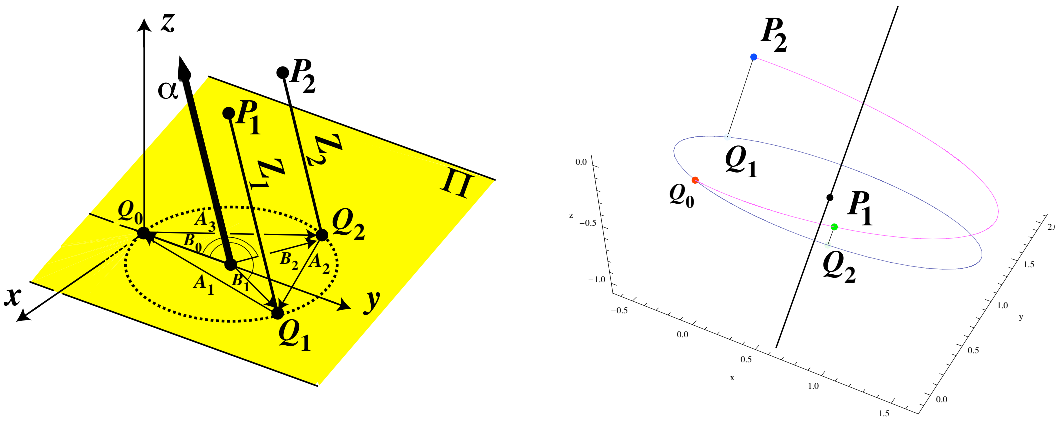

Figure 4. Right: The three points $\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2$ are projected on a plane Π through the origin and perpendicular to the vector $\boldsymbol{\alpha}$. The projected points are labeled $\mathbf{Q}_0, \mathbf{Q}_1, \mathbf{Q}_2$. Note that $\mathbf{P}_0$ lies at the origin and hence is $\mathbf{Q}_0 = \mathbf{P}_0$. Right: A helix passing through the points $(0,0,0)$, $(1,0,0)$, and $(-1/2,3/2,0)$ with $(k_1,k_2)=(0,0)$. The projected points and the projected circle are also shown. The projection vector (and helix axis) is $\boldsymbol{\alpha} \simeq (0.12, 0.47, 0.88)$ which means $\xi \simeq 0.50$ and $\eta \simeq 1.32$ radians, the radius is $R \simeq 1.12$.

• **Step 1: Projection.** First, we compute the projection on the plane, that is $\mathbf{Q}_i = \mathbf{P}_i - Z_i\boldsymbol{\alpha}$ with $Z_i = \boldsymbol{\alpha}\cdot\mathbf{P}_i$. where $\boldsymbol{\alpha} = (\alpha_1,\alpha_2,\alpha_3) = (\sin\xi\cos\eta, \sin\xi\sin\eta, \cos\xi)$ is the unit vector perpendicular to the projection plane Π passing through the origin.

• **Step 2: Projected circle.** Second, we compute the radius R and center $\mathbf{C}$ of the circle in the plane Π passing through the points $\mathbf{Q}_0, \mathbf{Q}_1, \mathbf{Q}_2$. To do so, we introduce the vectors $\mathbf{A}_1 = \mathbf{Q}_0 - \mathbf{Q}_1, \mathbf{A}_2 = \mathbf{Q}_1 - \mathbf{Q}_2, \mathbf{A}_3 = \mathbf{Q}_2 - \mathbf{Q}_0$ and in the case where none of the norm of these vectors vanishes, we obtain

$$R = \frac{|\mathbf{A}_1||\mathbf{A}_2||\mathbf{A}_3|}{2|\mathbf{A}_1 \times \mathbf{A}_2|}, \tag{26}$$

and

$$\mathbf{C} = \frac{-1}{2|\mathbf{A}_1 \times \mathbf{A}_2|^2} \cdot \left(|\mathbf{A}_3|^2 (\mathbf{A}_1 \cdot \mathbf{A_2}) \mathbf{Q}_1 + |\mathbf{A}_1|^2 (\mathbf{A}_2 \cdot \mathbf{A_3}) \mathbf{Q}_2 \right). \tag{27}$$

The case where one of the norm of $\mathbf{A}_i$ vanishes has to be dealt separately.

• **Step 3: Cylindrical coordinates.** Once the center and radius are known as a function of the vector $\boldsymbol{\alpha}$, we can compute the cylindrical coordinates of the three points $\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2$ with respect to an axis of orientation $\boldsymbol{\alpha}$ passing through the center $\mathbf{C}$. The Z coordinates of each points are $Z_i = \boldsymbol{\alpha}\cdot\mathbf{P}_i$, and the remaining missing coordinate is the angle φ. Let $\mathbf{B}_0 = \mathbf{Q}_0 - \mathbf{C}, \mathbf{B}_1 = \mathbf{Q}_1 - \mathbf{C}, \mathbf{B}_2 = \mathbf{Q}_2 - \mathbf{C}$ be the vectors from the center of the projected circle to the projected points. The cylindrical angles φ_i are the the angles between the vectors $\mathbf{B}_0$ and $\mathbf{B}_i$. Computationally, it is convenient to introduce the following quantities

$$u_1 = \frac{1}{R^2}\mathbf{B}_0 \cdot \mathbf{B}_1, \quad v_1 = \frac{\boldsymbol{\alpha}}{R^2} \cdot (\mathbf{B}_0 \times \mathbf{B}_1), \tag{28}$$

$$u_2 = \frac{1}{R^2}\mathbf{B}_0 \cdot \mathbf{B}_2, \quad v_2 = \frac{\boldsymbol{\alpha}}{R^2} \cdot (\mathbf{B}_0 \times \mathbf{B}_2). \tag{29}$$

Then the cylindrical angles are

$$\varphi_1 = \arg(u_1 + iv_1), \quad \varphi_2 = \arg(u_2 + iv_2), \tag{30}$$

where $\arg(z)$ was defined in Eq. (24).

• **Step 4: Colinearity condition**. Once the cylindrical coordinates are known, the condition that the three points lie on a helix is simply expressed as

$$(\varphi_2 + 2k_2\pi)Z_1 - (\varphi_1 + 2k_1\pi)Z_2 = 0. \tag{31}$$

This is, again, a condition on the two arbitrary angles which, together with (26), form for each (k_1, k_2) a system of 2 transcendental equations for 2 unknowns. In general, this systems needs to be solved numerically.

• **Step 5: The helix**. Once the values of $\boldsymbol{\alpha}$ have determined for a given pair (k_1, k_2) and a radius R, the equation for the helix is simply

$$\mathbf{r}(s) = (\cos(\delta s) - 1)\,\mathbf{B}_0 + \sin(\delta s)\,(\boldsymbol{\alpha} \times \mathbf{B}_0) + P\delta s\ \boldsymbol{\alpha}, \tag{32}$$

where $P = Z_1/(\varphi_1 + 2k_1\pi)$, and $\delta = 1/\sqrt{P^2 + R^2}$.

4.3 An example

We consider the points

$$\mathbf{P}_0 = (0,0,0), \quad (33)$$
$$\mathbf{P}_1 = (1,0,0), \quad (34)$$
$$\mathbf{P}_2 = (-1/2, 3/2, 0). \quad (35)$$

First, we compute the projection on the plane Π, that is

$$\mathbf{Q}_0 = (0,0,0),$$
$$\mathbf{Q}_1 = (1-\alpha_1^2, -\alpha_1\alpha_2, -\alpha_1\alpha_3),$$
$$\mathbf{Q}_2 = (-1/2 + (1/2\,\alpha_1 - 3/2\,\alpha_2)\alpha_1, 3/2 + (1/2\,\alpha_1 - 3/2\,\alpha_2)\,\alpha_2, (1/2\,\alpha_1 - 3/2\,\alpha_2)\,\alpha_3).$$

An example of a projection is given in Fig. 4

4.3.1 Minimal radius

We start by finding a lower bound on the radius by computing the cylinder passing through the 3 given points with minimal radius. Two cases are possible, either the 3 points are projected onto 2 points in the plane Π, in which case, the smallest radius is simply half the distance between the points, or the 3 points are projected to 3 distinct points. For the three points given, a direct computation shows that the minimal radius is obtained when $\mathbf{Q}_1 = \mathbf{Q}_2$, in which case: $\boldsymbol{\alpha} = (\mathbf{P}_2 - \mathbf{P}_1)/|\mathbf{P}_2 - \mathbf{P}_1| = (-\sqrt{2}/2, \sqrt{2}/2, 0)$ which implies a minimal radius $R = \sqrt{2}/4 \simeq 0.35$. An example of a helix with a radius $R = 0.41$ close to the minimal radius is given in Fig. 5.

4.3.2 Fixed radius with two helices

Next, we fix the radius to $R = 3/2$ and look for helices by solving the coupled system (26) and (31) for a given pair (k_1, k_2). For $(k_1, k_2) = (0,0)$, we find two different helices (See Fig. 6). For $(k_1, k_2) = (0,1)$ we also find two different helices, as well as in the case $(k_1, k_2) = (1,0)$. These four helices are drawn in Fig. 7.

4.3.3 Large radius

Finally, we also compute helices with large radii. Following the Proof of Proposition 6, we look for helices with $(k_1, k_2) = (0,1)$, $\xi = \frac{\pi}{2} - \epsilon$ and $\eta = \frac{\pi}{2} - A\epsilon$, where $A = \frac{b^2}{a(a-1)\pi(k_2+1)} = \frac{3}{2\pi}$. An example of helices with $R = 13$ ($\epsilon \simeq 0.02$), $R = 31$ ($\epsilon \simeq 0.008$), and $R = 131$ ($\epsilon \simeq 0.002$) is given in Fig. 8.

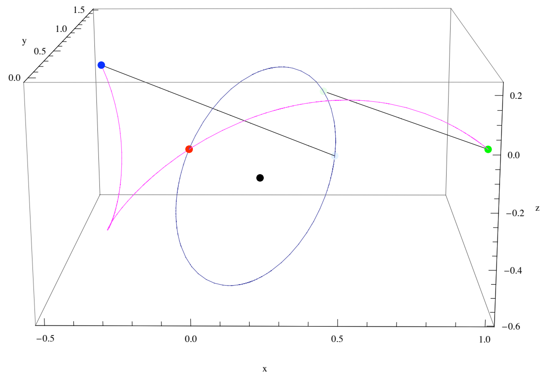

Figure 5. A helix passing through the points $(0,0,0)$, $(1,0,0)$, and $(-1/2,3/2,0)$ with a radius R=0.41, close to the minimal radius. The projection vector (and helix axis) is $\boldsymbol{\alpha} \simeq (-0.74, 0.66, 0.13)$.

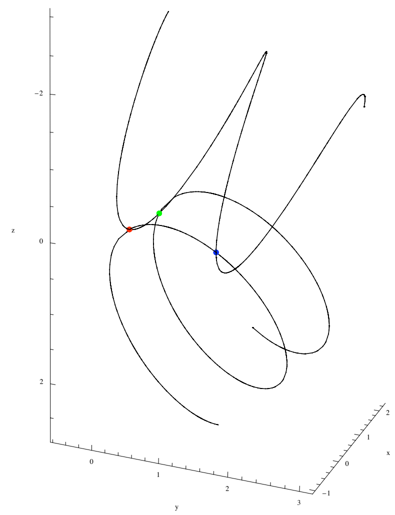

Figure 6. The two helices passing through the points $(0,0,0)$, $(1,0,0)$, and $(-1/2,3/2,0)$ with a radius $R = 3/2$ and $(k_1,k_2) = (0,0)$.

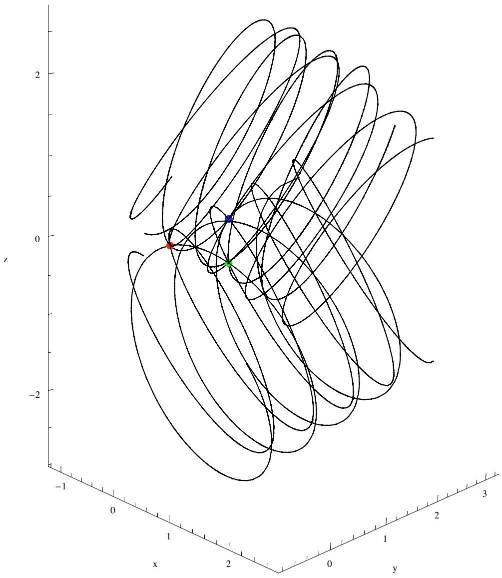

Figure 7. Four helices passing through the points $(0,0,0)$, $(1,0,0)$, and $(-1/2, 3/2, 0)$ with a radius $R = 3/2$. Two helices are with $(k_1, k_2) = (0,1)$ and two with $(k_1, k_2) = (1,0)$.

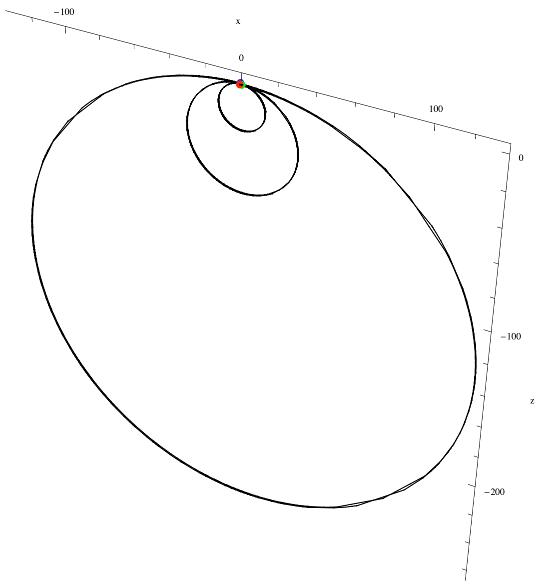

Figure 8. Helices through the points $(0,0,0)$, $(1,0,0)$, and $(-1/2, 3/2, 0)$ with $(k_1, k_2) = (0,1)$ and radii $R = 13$, $R = 31$, and $R = 131$.

5 Conclusion

The problem of identifying a countable family of helices passing through n points is an interesting and fundamental problem of classical geometry which has not been studied before. Here we have shown that, in general, no helix passes through 4 points and that there exist a continuous family of helices through 3 points. To identify a countable family of helices, an extra property of the helix, such as radius, pitch, torsion, or curvature, has to be specified. Here, we studied in more detail the case of 3 points and a given radius and show that for sufficiently large radii, one can always find helices through 3 points. A similar study for helices with given pitch, curvature, or torsion would follow the same steps. We conclude with two open problems:

Given n points in space, what is the condition for them to lie on a helix? Given 5 points or more, one can always construct all possible cylinders on which the points must lie and test, on each of these cylinders, whether a helix passes through the points. The case of 4 points is more complicated. Proposition 3 offers a partial answer to this problem. Considering 4 points at a time, one can construct the supporting cylinder and test whether there is a helix through these points. However, there may be a countable set of such cylinders which may not be tractable analytically.

Given three points in space, what is the *shortest helical path* through the points? By performing a rotation, a translation, and a dilation, it is always possible to find a reference frame in which the coordinates of the points are: $\mathbf{P}_0 = (0,0,0)$, $\mathbf{P}_1 = (1,0,0)$, and $\mathbf{P}_2 = (a,b,0)$ with a,b real numbers. An extensive numerical investigation of the shortest helical path for many values of a and b reveals that the arc of circle connecting the points is the shortest helical path. Therefore, we conjecture that the shortest helical arc through 3 points in space is the circular arc. An elegant proof, or a counter-example, of this conjecture would be valuable.

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