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The anelastic Ericksen problem: universal eigenstrains and deformations in compressible isotropic elastic solids

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The elastic Ericksen problem consists of finding deformations in isotropic hyperelastic solids that can be maintained for arbitrary strain-energy density functions. In the compressible case, Ericksen showed that only homogeneous deformations are possible. Here, we solve the anelastic version of the same problem, that is, we determine both the deformations and the eigenstrains such that a solution to the anelastic problem exists for arbitrary strain-energy density functions. Anelasticity is described by finite eigenstrains. In a nonlinear solid, these eigenstrains can be modelled by a Riemannian material manifold whose metric depends on their distribution. In this framework, we show that the natural generalization of the concept of homogeneous deformations is the notion of *covariantly homogeneous deformations*—deformations with covariantly constant deformation gradients. We prove that these deformations are the only universal deformations and that they put severe restrictions on possible *universal eigenstrains*. We show that, in a simply-connected body, for any distribution of universal eigenstrains the material manifold is a symmetric Riemannian manifold and that in dimensions 2 and 3 the universal eigenstrains are zero-stress.

1. Introduction

A *universal* or *controllable* deformation is one that is possible in every member of a class of materials in the absence of body forces. In other words, given a class of materials one can produce a universal deformation of a body made of any material in the class by applying only surface tractions. In the case of (unconstrained) compressible isotropic elastic solids, Ericksen [1] showed in a seminal paper that the only universal deformations are homogeneous deformations. The constrained case is more involved [2]. For instance, in the case of incompressible isotropic solids Ericksen [3] found four families of universal deformations. Later on, a fifth family was discovered independently by Singh & Pipkin [4] and Klingbeil & Shield [5]. Yet, Ericksen's problem in the case of incompressible isotropic solids has not been completely solved to this date as the case of deformations with constant principal invariants is still open. Ericksen's problem has played a central organizing role in the development of classical nonlinear elasticity as it has allowed the systematic analyses of elastic problems in various geometries.

Here, we generalize Ericksen's problem for compressible isotropic solids with finite eigenstrains in a simply-connected body. Finite eigenstrains [6] typically arise from anelastic effects and are particularly important in a number of applications, e.g. defects [7–9], temperature changes [10,11], bulk growth [12–14] and swelling [15,16].

At the heart of the problem is the notion of *strain*. In nonlinear elasticity, a strain is any measure of deformation. Having a measure of strain one can calculate the length of an infinitesimal line element in the deformed (current) configuration assuming one knows the length of this line element in its stress-free (initial) configuration. Choosing a measure of strain one would have a thermodynamically conjugate stress. Classical examples are: the deformation gradient and the first Piola–Kirchhoff stress $(\mathbf{F}, \mathbf{P})$ and the right Cauchy–Green strain and the second Piola–Kirchhoff stress $(\mathbf{C}, \mathbf{S})$. If we consider a general deformation that may involve both elastic and non-elastic components, a non-zero strain for the full deformation does not imply a non-zero stress at the same point. The part of strain that is related to stress through constitutive equations is called the *elastic strain*. The remaining part of strain is usually called *anelastic strain* or *eigenstrain*. Here, we use the term *eigenstrain* as coined by Mura [17]. Other terms for the same concept are: *pre-strain*, *initial strain* [18], *inherent strain* [19] and *transformation strain* [20].

In the absence of eigenstrains, for compressible isotropic hyperelastic materials the only universal deformations are homogeneous deformations. The extension to anelasticity naturally raises two fundamental questions:

(i) Do universal deformations exist in the presence of eigenstrains?
(ii) What is the class of eigenstrains for which these universal deformations exist?

For the Ericksen problem, we will refer to an eigenstrain field as *universal* for given universal deformations in a given class of materials if these deformations exist in the presence of the eigenstrain field.

It has been known since the seminal works of Eckart [21] and Kondo [18] that the inclusion of eigenstrains in anelasticity can be understood geometrically. The basic idea is to extend the traditional view of nonlinear elasticity by realizing that the effect of eigenstrains is to change the geometry of the reference (material) manifold from a Euclidean manifold to a Riemannian manifold. The metric of this Riemannian manifold depends explicitly on the eigenstrain distribution. These ideas are the starting point of a large body of the literature connecting geometric properties to anelasticity (see [22,23] and references therein). It enriches the traditional view of mechanics by providing a geometric interpretation to its fundamental equations. For instance, the traditional homogeneous deformations—deformations with constant deformation gradient—are not adequately defined when considering the deformation of a body with a non-Euclidean material manifold. However, it can be easily generalized to *covariantly homogeneous deformations* which play a key role in generalizing Ericksen's problem.

Here, we restrict our analysis to compressible isotropic solids (isotropic in the absence of eigenstrains) and characterize the universal eigenstrain fields and universal deformations of pre-strained compressible isotropic solids using Riemannian geometry. In particular, we show that the only universal deformations are the covariantly homogeneous deformations and the universal eigenstrains are zero-stress.

This paper is organized as follows. In §2, we briefly review basic concepts in local and global Riemannian geometry and the geometric formulation of anelasticity. In §3, we introduce the notion of universal eigenstrains through a simple example and then formulate the problem in its full generality. In §4 and 5, we characterize the universal deformations and eigenstrains in both dimensions 2 and 3 for (unconstrained) compressible isotropic (in their eigenstrain-free state) solids. Conclusions are given in §6.

2. Riemannian geometry and nonlinear anelasticity

We start with a short summary of the basic concepts of differential geometry, holonomy groups and symmetric Riemannian manifolds following Marsden & Hughes [24], Besse [25] and Joyce [26]. We then briefly review geometric anelasticity and discuss finite eigenstrains in nonlinear elastic solids that are isotropic in the absence of eigenstrains. We derive the form of the energy function for such solids when finite eigenstrains are present.

(a) Differential geometry

Given a smooth n-manifold $\mathcal{B}$, the tangent space to $\mathcal{B}$ at a point $X \in \mathcal{B}$ is denoted by $T_X\mathcal{B}$. Let $\mathcal{S}$ be another n-manifold and $\varphi : \mathcal{B} \to \mathcal{S}$ be a smooth and invertible map. A smooth vector field $\mathbf{W}$ on $\mathcal{B}$ assigns a vector $\mathbf{W}_X$ to every $X \in \mathcal{B}$ such that $X \mapsto \mathbf{W}_X \in T_X\mathcal{B}$ varies smoothly. For a vector field $\mathbf{W}$ on $\mathcal{B}$, $\varphi_*\mathbf{W} = T\varphi \cdot \mathbf{W} \circ \varphi^{-1}$ is a vector field on $\varphi(\mathcal{B})$ that is called the push-forward of $\mathbf{W}$ by φ. If $\mathbf{w}$ is a vector field on $\varphi(\mathcal{B}) \subset \mathcal{S}$, then $\varphi^*\mathbf{w} = T(\varphi^{-1}) \cdot \mathbf{w} \circ \varphi$ is a vector field on $\mathcal{B}$—the pullback of $\mathbf{w}$ by φ. The push-forward and pullback of vectors have the following coordinate representations: $(\varphi_*\mathbf{W})^a = F^a{}_A W^A$, $(\varphi^*\mathbf{w})^A = (F^{-1})^A{}_a w^a$, where $F^a{}_A = (T\varphi)^a{}_A$.

A type $\binom{0}{2}$-tensor at $X \in \mathcal{B}$ is a bilinear map $\mathbf{T} : T_X\mathcal{B} \times T_X\mathcal{B} \to \mathbb{R}$. In a local coordinate chart $\{X^A\}$ for $\mathcal{B}$, one has $\mathbf{T}(\mathbf{U}, \mathbf{W}) = T_{AB}U^A W^B$, where $\mathbf{U}, \mathbf{W} \in T_X\mathcal{B}$. We define an inner product $\mathbf{G}_X$ on the tangent space $T_X\mathcal{B}$ that varies smoothly, in the sense that if $\mathbf{U}$ and $\mathbf{W}$ are vector fields on $\mathcal{B}$, then

$$X \mapsto \mathbf{G}_X(\mathbf{U}_X, \mathbf{W}_X) =: \langle\!\langle \mathbf{U}_X, \mathbf{W}_X \rangle\!\rangle_{\mathbf{G}_X} \tag{2.1}$$

is a smooth function. Equipped with the metric $\mathbf{G}$, a smooth manifold $\mathcal{B}$ is a Riemannian manifold, denoted by $(\mathcal{B}, \mathbf{G})$. The metric $\mathbf{G}$ induces inner products (and hence norms) on tensor fields [27]. For example, for $\binom{0}{2}$-tensors $\mathbf{T}$ and $\mathbf{S}$, $\langle\!\langle \mathbf{T}_X, \mathbf{S}_X \rangle\!\rangle_{\mathbf{G}_X} = T_{AB}S_{MN}G^{AM}G^{BN}$.

Consider the two Riemannian manifolds $(\mathcal{B}, \mathbf{G})$ and $(\mathcal{S}, \mathbf{g})$ and a diffeomorphism (smooth map with smooth inverse) $\varphi : \mathcal{B} \to \mathcal{S}$. Then the push-forward of the metric $\mathbf{G}$ is a metric $\varphi_*\mathbf{G}$ on $\varphi(\mathcal{B}) \subset \mathcal{S}$ defined as

$$(\varphi_*\mathbf{G})_{\varphi(X)}(\mathbf{u}_{\varphi(X)}, \mathbf{w}_{\varphi(X)}) := \mathbf{G}_X((\varphi^*\mathbf{u})_X, (\varphi^*\mathbf{w})_X). \tag{2.2}$$

In components, $(\varphi_*\mathbf{G})_{ab} = (F^{-1})^A{}_a (F^{-1})^B{}_b G_{AB}$. Similarly, the pullback of the metric $\mathbf{g}$ is a metric in $\varphi^{-1}(\mathcal{S})$ that is denoted by $\varphi^*\mathbf{g}$ and is defined as

$$(\varphi^*\mathbf{g})_X(\mathbf{U}_X, \mathbf{W}_X) := \mathbf{g}_{\varphi(X)}((\varphi_*\mathbf{U})_{\varphi(X)}, (\varphi_*\mathbf{W})_{\varphi(X)}). \tag{2.3}$$

In components, $(\varphi^*\mathbf{g})_{AB} = F^a{}_A F^b{}_B g_{ab}$. If $\mathbf{g} = \varphi_*\mathbf{G}$ (or, equivalently, $\mathbf{G} = \varphi^*\mathbf{g}$), φ is called an isometry and the Riemannian manifolds $(\mathcal{B}, \mathbf{G})$ and $(\mathcal{S}, \mathbf{g})$ are isometric. An isometry, by definition, preserves distances. Suppose $\mathbf{Q}$ and $\mathbf{R}$ are two-point tensors. Their inner product is defined as $\langle\!\langle \mathbf{Q}_X, \mathbf{R}_X \rangle\!\rangle_{\mathbf{G}_X, \mathbf{g}_x} = Q^a{}_A R^b{}_B g_{ab} G^{AB}$.

In a local coordinate chart $\{X^A\}$ for $(\mathcal{B}, \mathbf{G})$ the Riemann curvature tensor has the components $\mathcal{R}^A{}_{BCD}$ defined as

$$\mathcal{R}^A{}_{BCD} = \frac{\partial \Gamma^A{}_{BD}}{\partial X^C} - \frac{\partial \Gamma^A{}_{BC}}{\partial X^D} + \Gamma^A{}_{CE}\Gamma^E{}_{BD} - \Gamma^A{}_{DE}\Gamma^E{}_{BC}. \tag{2.4}$$

Obviously, $\mathcal{R}^A{}_{BDC} = -\mathcal{R}^A{}_{BCD}$. A flat Riemannian manifold has an identically vanishing Riemann curvature tensor. For a vector field $\mathbf{W}$ with components W^A the Ricci identity reads

$$W^A{}_{|BC} - W^A{}_{|CB} = \mathcal{R}^A{}_{BCD} W^D. \tag{2.5}$$

For a 1-form $\boldsymbol{\alpha}$ with components α_A, $\alpha_A W^A$ is a scalar and hence $(\alpha_A W^A)_{|BC} = (\alpha_A W^A)_{|CB}$. Expanding this and using the Ricci identity for $\mathbf{W}$ one obtains the following Ricci identity for a 1-form

$$\alpha_{A|BC} - \alpha_{A|CB} = \mathcal{R}^D{}_{BCA} \alpha_D. \tag{2.6}$$

The Ricci curvature $\mathbf{R}$ is defined as $R_{CD} = \mathcal{R}^A{}_{ACD}$ and is symmetric. The scalar curvature is defined as $R = \mathrm{tr}_{\mathbf{G}} \mathbf{R} = R_{AB} G^{AB}$. In dimensions 2 and 3, the Ricci curvature fully determines the Riemannian curvature [28]. In dimension 2, the Ricci curvature can be written as $R_{AB} = K G_{AB}$, where $K = \frac{1}{2} R$ is the Gaussian curvature. For a Riemannian n-manifold ($n > 2$), $(\mathcal{B}, \mathbf{G})$ is an Einstein manifold if $R_{AB} = K G_{AB}$, where K is a function on $\mathcal{B}$.

Next, we define Riemannian product manifolds [26]. Let $(\mathcal{B}_1, \mathbf{G}_1)$ and $(\mathcal{B}_2, \mathbf{G}_2)$ be Riemannian manifolds and $\mathcal{B}_1 \times \mathcal{B}_2$ their product manifold. At each point $(X_1, X_2) \in \mathcal{B}_1 \times \mathcal{B}_2$, we have the following direct sum $T_{(X_1,X_2)}(\mathcal{B}_1 \times \mathcal{B}_2) \cong T_{X_1}\mathcal{B}_1 \oplus T_{X_2}\mathcal{B}_2$, where $\cong$ means 'isomorphic to'. The product metric $\mathbf{G}_1 \times \mathbf{G}_2$ on $\mathcal{B}_1 \times \mathcal{B}_2$ is defined as

$$\mathbf{G}_1 \times \mathbf{G}_2|_{(X_1,X_2)} = \mathbf{G}_1|_{X_1} + \mathbf{G}_2|_{X_2}, \quad \forall\, X_1 \in \mathcal{B}_1, \;\; X_2 \in \mathcal{B}_2. \tag{2.7}$$

The Riemannian manifold $(\mathcal{B}_1 \times \mathcal{B}_2, \mathbf{G}_1 \times \mathbf{G}_2)$ is called a Riemannian product space. A Riemannian manifold that is isometric to a Riemannian product space is called reducible (decomposable). Otherwise, it is irreducible (indecomposable). Note that, for a Riemannian product space, $\nabla^{\mathbf{G}_1 \times \mathbf{G}_2}_{(\mathbf{U}_1,\mathbf{U}_2)}(\mathbf{W}_1, \mathbf{W}_2) = (\nabla^{\mathbf{G}_1}_{\mathbf{U}_1}\mathbf{W}_1, \nabla^{\mathbf{G}_2}_{\mathbf{U}_2}\mathbf{W}_2)$. In particular, the Ricci curvature of the Riemannian product space is written as

$$\mathbf{R}((\mathbf{U}_1, \mathbf{U}_2), (\mathbf{W}_1, \mathbf{W}_2)) = \mathbf{R}_1(\mathbf{U}_1, \mathbf{W}_1) + \mathbf{R}_2(\mathbf{U}_2, \mathbf{W}_2). \tag{2.8}$$

Let $(\mathcal{B}, \mathbf{G})$ be a connected Riemannian manifold. Using the Levi–Civita connection $\nabla^{\mathbf{G}}$ (the unique connection that is torsion-free and is compatible with the metric) one can define parallel transport of vectors along a curve. Suppose $\gamma : [0, 1] \to \mathcal{B}$ is a curve and $\mathbf{W}_0 \in T_{\gamma(0)}\mathcal{B}$. The parallel transport of $\mathbf{W}_0$ along γ is the unique vector field $\mathbf{W}$ such that $\nabla^{\mathbf{G}}_{\gamma'}\mathbf{W} = \mathbf{0}$. If γ is a closed curve (a loop) based at $X = \gamma(0) = \gamma(1)$, the parallel transport map $P_\gamma : T_{\gamma(0)}\mathcal{B} \to T_{\gamma(1)}\mathcal{B}$ is an endomorphism (a homomorphism from the tangent space to itself). For the composition of loops one has $P_{\gamma_1 \circ \gamma_2} = P_{\gamma_2} P_{\gamma_1}$. The holonomy group $\mathrm{Hol}_X(\mathbf{G})$ based at X is defined as

$$\mathrm{Hol}_X(\mathbf{G}) = \{P_\gamma : \gamma \text{ is a loop based at } X\}. \tag{2.9}$$

For a connected manifold $\mathrm{Hol}_X(\mathbf{G})$ is independent of the base point and one simply can write $\mathrm{Hol}(\mathbf{G})$ for the holonomy group of a connected Riemannian manifold $(\mathcal{B}, \mathbf{G})$. The restricted holonomy group is defined as

$$\mathrm{Hol}^0_X(\mathbf{G}) = \{P_\gamma : \gamma \text{ is a null-homotopic loop based at } X\}, \tag{2.10}$$

where a null-homotopic loop based at X is a loop that can be continuously shrunk to X. If $\mathcal{B}$ is simply connected, $\mathrm{Hol}(\mathbf{G}) = \mathrm{Hol}^0(\mathbf{G})$. Suppose $\mathbf{T}$ is a covariantly constant tensor field, i.e. $\nabla^{\mathbf{G}}\mathbf{T} = \mathbf{0}$. Then $\mathbf{T}(X)$ is fixed by the action of $\mathrm{Hol}_X(\mathbf{G})$ because, given any loop, $\mathbf{T}$ is parallel along the loop. Conversely, if $\mathbf{T}(X)$ is fixed by the action of $\mathrm{Hol}_X(\mathbf{G})$, there exists a unique tensor field $\mathbf{T}$ such that $\nabla^{\mathbf{G}}\mathbf{T} = \mathbf{0}$ and $\mathbf{T}|_X = \mathbf{T}(X)$. In a Riemannian manifold $\nabla^{\mathbf{G}}\mathbf{G} = \mathbf{0}$, i.e. $\mathbf{G}$ is a covariantly constant tensor field and hence at any point $X \in \mathcal{B}$ it is invariant under the action of $\mathrm{Hol}_X(\mathbf{G})$. However, we know that the group of transformations of $T_X\mathcal{B}$ preserving the metric is the orthogonal group $O(n)$. Therefore, $\mathrm{Hol}(\mathbf{G})$ is a subgroup of $O(n)$. It can be shown that $\mathrm{Hol}^0(\mathbf{G})$ is a connected Lie subgroup of $SO(n)$.

A Riemannian manifold $(\mathcal{B}, \mathbf{G})$ is homogeneous if the group of isometries $\mathrm{Iso}(\mathcal{B}, \mathbf{G})$ acts transitively, i.e. for any $X_1, X_2 \in \mathcal{B}$, there is an isometry ϕ such that $\phi(X_1) = X_2$. In a homogeneous space the curvature tensor is covariantly constant, i.e. $\nabla^{\mathbf{G}}\boldsymbol{\mathcal{R}} = \mathbf{0}$. $(\mathcal{B}, \mathbf{G})$ is a Riemannian symmetric space if for each $X \in \mathcal{B}$ there is an isometry ϕ_X such that $\phi_X(X) = X$ and $T_X(\phi_X) = -Id_{T_X\mathcal{B}}$. That is, Riemannian symmetric spaces are those Riemannian manifolds that have point reflections. It can be shown that any Riemannian symmetric space is homogeneous.

Berger [29] showed that for a simply-connected manifold $\mathcal{B}$, if $(\mathcal{B}, \mathbf{G})$ is a non-symmetric Riemannian manifold, and is irreducible, there are only seven possibilities for the holonomy group. It is known that any simply-connected Riemannian symmetric space is the Riemannian product of a Euclidean space and a finite number of irreducible Riemannian symmetric spaces [25, p.194]. It is also known that a simply-connected and irreducible Riemannian symmetric space is Einstein [30,31].

A vector field with vanishing covariant derivative plays a particularly important role in our analysis of universal eigenstrains. The problem is to identify which Riemannian manifolds support such non-trivial vector fields. This problem is closely related to holonomy groups. For an orientable manifold $\mathrm{Hol}(\mathbf{G}) \subset SO(n)$. The Riemannian manifold $(\mathcal{B}, \mathbf{G})$ is flat if and only if $\mathrm{Hol}^0(\mathbf{G})$ is reduced to the identity (trivial group). For a generic Riemannian manifold $\mathrm{Hol}^0(\mathbf{g}) = SO(n)$. In this case, there are no differential forms with vanishing covariant derivative. As a matter of fact, there are 'very few Riemannian manifolds which admit a nontrivial exterior form with zero covariant derivative' [25, p. 306]. We will see in §4 that in the case of 1-forms these special manifolds are closely related to universal deformations. In particular, we will use the fact that for $(\mathcal{B}, \mathbf{G})$ a Riemannian 3-manifold with a global covariantly constant 1-form $\boldsymbol{\alpha}$, which is not identically zero, $(\mathcal{B}, \mathbf{G})$ must be a symmetric Riemannian manifold. This is a direct application of [25, corollary 10.110].

(b) Geometric elasticity and finite eigenstrains

A body $\mathcal{B}$ is identified with a Riemannian manifold $(\mathcal{B}, \mathbf{G})$ and a deformation of $\mathcal{B}$ is a mapping $\varphi : \mathcal{B} \to \mathcal{S}$, where $(\mathcal{S}, \mathbf{g})$ is another Riemannian manifold—the ambient space. It is assumed that the body is stress free in the material manifold. The material velocity is the map $\mathbf{V}_t : \mathcal{B} \to T_{\varphi_t(\mathbf{X})}\mathcal{S}$ given by $\mathbf{V}_t(\mathbf{X}) = \mathbf{V}(\mathbf{X}, t) = \partial\varphi(\mathbf{X}, t)/\partial t$. The deformation gradient is the tangent map of φ that is denoted by $\mathbf{F} = T\varphi$. At each point $\mathbf{X} \in \mathcal{B}$, this is a linear map $\mathbf{F}(\mathbf{X}) : T_\mathbf{X}\mathcal{B} \to T_{\varphi(\mathbf{X})}\mathcal{S}$. If $\{x^a\}$ and $\{X^A\}$ are local coordinate charts on $\mathcal{S}$ and $\mathcal{B}$, respectively, the components of $\mathbf{F}$ are written as $F^a{}_A(\mathbf{X}) = (\partial\varphi^a/\partial X^A)(\mathbf{X})$. The transpose of $\mathbf{F}$ is defined by $\mathbf{F}^\mathsf{T} : T_\mathbf{x}\mathcal{S} \to T_\mathbf{X}\mathcal{B}$, $\langle\!\langle \mathbf{FV}, \mathbf{v} \rangle\!\rangle_\mathbf{g} = \langle\!\langle \mathbf{V}, \mathbf{F}^\mathsf{T}\mathbf{v} \rangle\!\rangle_\mathbf{G}$, for all $\mathbf{V} \in T_\mathbf{X}\mathcal{B}$, $\mathbf{v} \in T_\mathbf{x}\mathcal{S}$. In components, $(F^\mathsf{T}(\mathbf{X}))^A{}_a = g_{ab}(\mathbf{x})F^b{}_B(\mathbf{X})G^{AB}(\mathbf{X})$. The two-point tensor $\mathbf{F}$ has the local representation $\mathbf{F} = F^a{}_A \frac{\partial}{\partial x^a} \otimes dX^A$. The right Cauchy–Green deformation tensor $\mathbf{C}(X) : T_\mathbf{X}\mathcal{B} \to T_\mathbf{X}\mathcal{B}$ is defined as $\mathbf{C}(\mathbf{X}) = \mathbf{F}(\mathbf{X})^\mathsf{T}\mathbf{F}(\mathbf{X})$, which in components reads $C^A{}_B = (F^\mathsf{T})^A{}_a F^a{}_B$. One can show that $\mathbf{C}^\flat = \varphi^*(\mathbf{g})$, i.e. $C_{AB} = (g_{ab} \circ \varphi)F^a{}_A F^b{}_B$. The governing equations of nonlinear elasticity then read

$$\frac{\partial \rho_0}{\partial t} = 0, \tag{2.11}$$

$$\mathrm{Div}\,\mathbf{P} + \rho_0\mathbf{B} = \rho_0\mathbf{A} \tag{2.12}$$

and

$$\mathbf{P}\mathbf{F}^\mathsf{T} = \mathbf{F}\mathbf{P}^\mathsf{T}, \tag{2.13}$$

where ρ_0 is the material mass density and $\mathbf{P}$ is the first Piola–Kirchhoff stress. The relation between $\mathbf{P}$ and the Cauchy stress $\boldsymbol{\sigma}$ is $J\sigma^{ab} = P^{aA}F^b{}_A$, where $J = \sqrt{\det\mathbf{g}/\det\mathbf{G}}\det\mathbf{F}$ is the Jacobian. The left Cauchy–Green deformation tensor $\mathbf{B}^\sharp = \varphi^*(\mathbf{g}^\sharp)$ has components $B^{AB} = (F^{-1})^A{}_a(F^{-1})^B{}_b g^{ab}$. The spatial analogues of $\mathbf{C}^\flat$ and $\mathbf{B}^\sharp$ are

$$\mathbf{c}^\flat = \varphi_*(\mathbf{G}), \quad c_{ab} = (F^{-1})^A{}_a(F^{-1})^B{}_b G_{AB}, \tag{2.14}$$

and

$$\mathbf{b}^\sharp = \varphi_*(\mathbf{G}^\sharp), \quad b^{ab} = F^a{}_A F^b{}_B G^{AB}, \tag{2.15}$$

$\mathbf{b}^\sharp$ is the Finger deformation tensor. The tensors $\mathbf{C}$ and $\mathbf{b}$ have the same principal invariants I_1, I_2 and I_3 [32]. Importantly, for an isotropic material the strain energy function W depends only on the principal invariants of $\mathbf{b}$.

We consider a stress-free body $\mathcal{B}$ embedded in the Euclidean ambient space. The flat metric of the body induced from that of the Euclidean ambient space is denoted by $\overset{\circ}{\mathbf{G}}$. The inner product induced by this metric is denoted by $\langle\!\langle , \rangle\!\rangle_{\overset{\circ}{\mathbf{G}}}$ so that for given vectors $d\overset{\circ}{X}, d\overset{\circ}{Y} \in T_X\mathcal{B}$, their inner product is $\langle\!\langle d\overset{\circ}{X}, d\overset{\circ}{Y} \rangle\!\rangle_{\overset{\circ}{\mathbf{G}}}$. In particular, the square of the length of the vector $d\overset{\circ}{X}$ is $\langle\!\langle d\overset{\circ}{X}, d\overset{\circ}{X} \rangle\!\rangle_{\overset{\circ}{\mathbf{G}}}$.

Suppose now that the same body is given a distribution of eigenstrains. We are not concerned with nucleation and/or dynamics of eigenstrains and simply assume that a static distribution of eigenstrains is given. The body with eigenstrains is, in general, residually stressed. If we let the infinitesimal line element $d\overset{\circ}{X}$ relax, it would transform to another vector dX. The linear transformation $\mathbf{K}(X) : T_X\mathcal{B} \to T_X\mathcal{B}$ such that $dX = \mathbf{K}d\overset{\circ}{X}$ is a measure of eigenstrains. Note that

$$\langle\!\langle d\overset{\circ}{X}, d\overset{\circ}{Y} \rangle\!\rangle_{\overset{\circ}{\mathbf{G}}} = \langle\!\langle \mathbf{K}d\overset{\circ}{X}, \mathbf{K}d\overset{\circ}{Y} \rangle\!\rangle_{\mathbf{K}_*\overset{\circ}{\mathbf{G}}} = \langle\!\langle dX, dY \rangle\!\rangle_{\mathbf{G}}, \tag{2.16}$$

where we call $\mathbf{G} = \mathbf{K}_*\overset{\circ}{\mathbf{G}}$ the material metric (note that $\mathbf{K}$ is a map that is defined locally). It is seen that the inner products and hence local angles and distances in $(\mathcal{B}, \overset{\circ}{\mathbf{G}})$ and $(\mathcal{B}, \mathbf{G})$ are identical. This identity implies that $(\mathcal{B}, \mathbf{G})$ is the stress-free material manifold of the body with eigenstrains.

Suppose that the body in the absence of eigenstrains is isotropic. This means that the energy function with respect to the manifold $(\mathcal{B}, \overset{\circ}{\mathbf{G}})$ depends only on the three principal invariants of the right Cauchy–Green tensor, i.e. $W = W(\overset{\circ}{I}_1, \overset{\circ}{I}_2, \overset{\circ}{I}_3)$. The problem is then to determine the form of the energy function in the presence of eigenstrains for isotropic materials. We show that in this case the energy function has the same form with respect to the material manifold $(\mathcal{B}, \mathbf{G})$, i.e. $W = W(I_1, I_2, I_3)$. Note that the principal invariants explicitly depend on the metric $\mathbf{G}$. For example, $I_1 = C_{AB}G^{AB}$. We start by recalling that a body is materially covariant if, for any diffeomorphism $\Xi : \mathcal{B} \to \mathcal{B}$, one has [24]

$$\Xi^*\{W(\mathbf{C}, \mathbf{G})\} = W(\Xi^*\mathbf{C}, \Xi^*\mathbf{G}). \tag{2.17}$$

Marsden & Hughes [24] proved that a body is materially covariant if and only if it is isotropic. Since the body is isotropic in the absence of eigenstrains, it is materially covariant. For a given deformation mapping $\varphi : \mathcal{B} \to \mathcal{S}$, and hence $\mathbf{C} = \varphi^*\mathbf{g}$, the energy function is written as $W = W(\mathbf{C}, \mathbf{G})$. However, knowing that $\mathbf{G} = \mathbf{K}_*\overset{\circ}{\mathbf{G}}$, we can write

$$W(\mathbf{C}, \mathbf{G}) = W(\mathbf{C}, \mathbf{K}_*\overset{\circ}{\mathbf{G}}) = W(\mathbf{K}^*\mathbf{C}, \overset{\circ}{\mathbf{G}}), \tag{2.18}$$

where we have used the fact that $\mathbf{K}$ is the tangent map of a local material diffeomorphism and that under this local diffeomorphism the energy function is covariant. Now, recall that the energy function for zero eigenstrain can be written as $W = W(\overset{\circ}{I}_1, \overset{\circ}{I}_2, \overset{\circ}{I}_3)$, where $\overset{\circ}{I}_i$ $(i = 1, 2, 3)$ are the principal invariants of $\mathbf{K}^*\mathbf{C}$ using the metric $\overset{\circ}{\mathbf{G}}$. However, these are identical to the principal invariants of $\mathbf{C}$ using the metric $\mathbf{G} = \mathbf{K}_*\overset{\circ}{\mathbf{G}}$. This proves our claim. In summary, in the presence of eigenstrains the energy function has the same form but the invariants of $\mathbf{C}$ are calculated using the metric $\mathbf{G}$.

Note that the map $\mathbf{K}(X) : T_X\mathcal{B} \to T_X\mathcal{B}$ is defined locally and is not necessarily the tangent map of any diffeomorphism $\Xi : \mathcal{B} \to \mathcal{B}$. However, the definition of a materially covariant solid is local and only depends on the fact that the energy function is a tensorial function of its arguments.

3. A simple example

We use a motivating example to illustrate Ericksen's problem for materials with eigenstrains. We consider an isotropic cylinder. In the absence of eigenstrains, it is well known that a uniform radial expansion or compression is a universal deformation. The question is to determine the axisymmetric universal deformations and their corresponding universal eigenstrains.

Here, we choose a restricted class of eigenstrains with the same radial dependence as the deformation mapping.

We consider an infinitely long solid cylinder that is only allowed to deform axisymmetrically (with no dependence on the Z-coordinate). That is, in cylindrical coordinates, we have $(r,\theta,z)=(r(R),\Theta,Z)$ with deformation gradient $\mathbf{F}=\mathrm{diag}\{r'(R),1,1\}$. The ambient space is the flat Euclidean space with the metric $\mathbf{g}=\mathrm{diag}\{1,r^2,1\}$. The eigenstrains are specified by four arbitrary functions of R so that the material metric has the form

$$\mathbf{G}=\begin{pmatrix}\xi^2(R) & \psi(R) & 0\\ \psi(R) & \eta^2(R) & 0\\ 0 & 0 & 1\end{pmatrix}, \tag{3.1}$$

where $\xi^2(R)\eta^2(R)-\psi^2(R)>0$. This specific form for the metric is chosen as the simplest non-trivial generalization of a flat metric in cylindrical geometry. We look for restrictions on these four functions so that the axially-symmetric deformations of the cylinder with the material metric (3.1) is universal. The Finger tensor and its inverse have the following representations in cylindrical coordinates:

$$\mathbf{b}=\begin{pmatrix}\dfrac{r'^2\eta^2}{\eta^2\xi^2-\psi^2} & -\dfrac{r'\psi}{\eta^2\xi^2-\psi^2} & 0\\ -\dfrac{r'\psi}{\eta^2\xi^2-\psi^2} & \dfrac{\xi^2}{\eta^2\xi^2-\psi^2} & 0\\ 0 & 0 & 1\end{pmatrix} \quad\text{and}\quad \mathbf{c}=\begin{pmatrix}\dfrac{\xi^2}{r'^2} & \dfrac{\psi}{r^2r'} & 0\\ \dfrac{\psi}{r^2r'} & \dfrac{\eta^2}{r^4} & 0\\ 0 & 0 & 1\end{pmatrix}. \tag{3.2}$$

The principal invariants of $\mathbf{b}$ read

$$I_1=\frac{\xi^2r^2}{\eta^2\xi^2-\psi^2}+\frac{\eta^2r'^2}{\eta^2\xi^2-\psi^2}+1,\quad I_2=I_1+I_3-1\quad\text{and}\quad I_3=\frac{r^2r'^2}{\eta^2\xi^2-\psi^2}. \tag{3.3}$$

The Cauchy stress has the following non-zero components:

$$\left.\begin{aligned}\sigma^{rr}&=\frac{r'}{r\sqrt{\xi^2\eta^2-\psi^2}}[\eta^2\alpha+(r^2+\eta^2)\beta+r^2\gamma],\\ \sigma^{\theta\theta}&=\frac{1}{rr'\sqrt{\xi^2\eta^2-\psi^2}}[\xi^2\alpha+(r'^2+\xi^2)\beta+r'^2\gamma],\\ \sigma^{zz}&=\frac{(\xi^2\eta^2-\psi^2)\alpha+(\xi^2r^2+\eta^2r'^2)\beta+r'^2r^2\gamma}{rr'\sqrt{\xi^2\eta^2-\psi^2}},\\ \sigma^{r\theta}&=-\frac{\psi(\alpha+\beta)}{r\sqrt{\xi^2\eta^2-\psi^2}},\end{aligned}\right\} \tag{3.4}$$

where

$$\alpha=2\frac{\partial W}{\partial I_1},\quad \beta=2\frac{\partial W}{\partial I_2}\quad\text{and}\quad \gamma=2\frac{\partial W}{\partial I_3}. \tag{3.5}$$

The only non-vanishing components of the Cauchy equation for the balance of linear momentum is the radial equilibrium equation

$$\frac{\partial\sigma^{rr}}{\partial r}+\frac{1}{r}\sigma^{rr}-r\sigma^{\theta\theta}=0, \tag{3.6}$$

and the circumferential equilibrium equation $\sigma^{\theta b}{}_{|b}=0$. For our particular problem, this last equation further simplifies to $r^3\sigma^{r\theta}=C$, where C is a constant. Thus

$$-\frac{r^2\psi}{\sqrt{\xi^2\eta^2-\psi^2}}(\alpha+\beta)=C. \tag{3.7}$$

For a deformation to be universal, it should exist for arbitrary energy functions. Therefore, this last relation must hold as α and β vary independently. Therefore, their coefficients must be zero and we have $\psi(R)=0$, which is the first restriction on the material metric. With this condition, the radial equilibrium equation (3.6) reads

$$\left[\frac{r'\eta}{\xi r}\alpha+\left(\frac{\eta r'}{\xi r}+\frac{rr'}{\xi\eta}\right)\beta+\frac{rr'}{\xi\eta}\gamma\right]'+r'\left(\frac{r'\eta}{r^2\xi}-\frac{\xi}{r'\eta}\right)(\alpha+\beta)=0. \tag{3.8}$$

If we assume first that $W=W(I_3)$, then $\alpha=\beta=0$ and we conclude that $I_3=r^2r'^2/\xi^2\eta^2$ is constant and hence

$$\frac{r^2r'^2}{\xi^2\eta^2}=C_3. \tag{3.9}$$

Next, if we assume $W=W(I_1)$, we conclude that I_1 is constant and hence

$$\frac{r^2}{\eta^2}+\frac{r'^2}{\xi^2}=C_1 \tag{3.10}$$

and also

$$\left(\frac{r'\eta}{\xi r}\right)'+r'\left(\frac{r'\eta}{r^2\xi}-\frac{\xi}{r'\eta}\right)=0. \tag{3.11}$$

Finally, assuming that $W=W(I_2)$, we obtain ($I_2=I_1+I_3-1$ is a constant)

$$\left(\frac{\eta r'}{\xi r}+\frac{rr'}{\xi\eta}\right)'+r'\left(\frac{r'\eta}{r^2\xi}-\frac{\xi}{r'\eta}\right)=0. \tag{3.12}$$

From (3.9) and (3.10), we observe that r'^2/ξ^2 and r^2/η^2 are constants and hence (3.11) is reduced to

$$\frac{r'}{r}\left(\frac{r'^2}{\xi^2}-\frac{r^2}{\eta^2}\right)=0, \tag{3.13}$$

that is,

$$\frac{r'^2}{\xi^2}=\frac{r^2}{\eta^2}=C^2, \tag{3.14}$$

and (3.12) is trivially satisfied. Therefore, we conclude that the deformation governed by $r=r(R)$ is universal with universal eigenstrains if

$$\xi(R)=\eta'(R)\quad\text{and}\quad r(R)=C\eta(R). \tag{3.15}$$

We see from this simple example that universal deformations and universal eigenstrains are intimately coupled. We also see that for anelastic problems a universal deformation for compressible materials does not need to be homogeneous (in the traditional sense of having a constant deformation gradient). Therefore, the notion of homogeneity must be generalized for such systems. To do so, we first note that the only non-zero Levi–Civita connection coefficients of $\mathbf{G}$ are

$$\Gamma^1{}_{11}=\frac{\xi'(R)}{\xi(R)},\quad \Gamma^1{}_{22}=-\frac{\eta(R)}{\xi(R)}\quad\text{and}\quad \Gamma^2{}_{12}=\Gamma^2{}_{21}=\frac{\xi(R)}{\eta(R)}. \tag{3.16}$$

For $\mathbf{g}$ the non-zero connection coefficients are $\gamma^1{}_{22}=-r(R)$ and $\gamma^2{}_{12}=\gamma^2{}_{21}=\frac{1}{r(R)}$. Using these, the only possible non-vanishing components of $\nabla^{\mathbf{G}}\mathbf{F}$ are

$$\left.\begin{aligned} F^1{}_{1|1}&=\frac{\partial F^a{}_1}{\partial R}-\Gamma^1{}_{11}F^1{}_1=r''(R)-\frac{\xi'(R)}{\xi(R)}r'(R),\\ F^1{}_{2|2}&=-\Gamma^1{}_{22}F^1{}_1+\gamma^1{}_{22}(F^2{}_2)^2=\frac{\eta(R)}{\xi(R)}r'(R)-r(R)\\ \text{and}\quad F^2{}_{1|2}&=F^2{}_{2|1}=-\Gamma^2{}_{12}F^2{}_2+\gamma^2{}_{12}F^1{}_1F^2{}_2=-\frac{\xi(R)}{\eta(R)}+\frac{r'(R)}{r(R)}. \end{aligned}\right\} \tag{3.17}$$

Using (3.15), all these components vanish identically. Hence, we conclude that *the deformation is covariantly homogeneous*, that is,

$$\nabla^{\mathbf{G}}\mathbf{F}=\mathbf{0}. \tag{3.18}$$

We will show that, in the general case, this property is the natural generalization of homogeneity for anelastic systems.

Many systems in the presence of eigenstrains develop residual stresses. It is, therefore, of interest to characterize the stresses corresponding to universal eigenstrains. In [9], we showed that, for the case of a compressible (or incompressible) cylinder, the residual stresses are identically zero if and only if $C\xi(R) = \eta'(R)$, for any constant C. These special eigenstrains are called zero-stress eigenstrains. In particular, this condition is verified when (3.15) holds and $C = 1$, and we conclude that, for our example, *the universal eigenstrains are zero-stress.*

4. Universal deformations are covariantly homogeneous

Next, we show that the following two properties observed in the above example hold for the general problem. Namely, we will prove that all the universal deformations and eigenstrains in compressible isotropic anelasticity are such that

— the universal deformations are covariantly homogeneous; and
— the universal eigenstrains are zero-stress.

Our strategy is to follow the same steps as in the example. We write the general equations provided to us by the balance of linear momentum and, assuming that they should hold for arbitrary energy functions, obtain an over-determined system of equations that constrains the possible form of the principal invariants and the Cauchy–Green strain. Then, the compatibility conditions are used to find simultaneous constraints on the deformation gradient and the eigenstrains.

Ericksen [1] assumed that both the reference configuration and the ambient space are Euclidean. Here we assume that the ambient space is Euclidean but that, in the presence of eigenstrains, the material manifold $(\mathcal{B}, \mathbf{G})$ is, in general, non-flat.

For a compressible isotropic solid the Cauchy stress has the following representation [33–35]:

$$\sigma^{ab} = \frac{1}{\sqrt{I_3}}[(I_2\beta + I_3\gamma)g^{ab} + \alpha b^{ab} - I_3\beta c^{ab}], \tag{4.1}$$

where

$$\alpha = 2\frac{\partial W}{\partial I_1}, \quad \beta = 2\frac{\partial W}{\partial I_2} \quad \text{and} \quad \gamma = 2\frac{\partial W}{\partial I_3}. \tag{4.2}$$

Since the ambient space is Euclidean, we can use a single Cartesian coordinate chart $\{x^a\}$, so that $g_{ab} = \delta_{ab}$, and we have

$$\sigma^{ab} = \frac{1}{\sqrt{I_3}}[(I_2\beta + I_3\gamma)\delta^{ab} + \alpha b^{ab} - I_3\beta c^{ab}]. \tag{4.3}$$

In the absence of body forces, the equilibrium equations $\mathrm{div}\boldsymbol{\sigma} = \mathbf{0}$, in Cartesian coordinates read $\sigma^{ab}{}_{,b} = 0$. Note that

$$\left.\begin{aligned}
\alpha_{,b} &= 2\frac{\partial^2 W}{\partial I_1^2}I_{1,b} + 2\frac{\partial^2 W}{\partial I_1 \partial I_2}I_{2,b} + 2\frac{\partial^2 W}{\partial I_1 \partial I_3}I_{3,b},\\
\beta_{,b} &= 2\frac{\partial^2 W}{\partial I_1 \partial I_2}I_{1,b} + 2\frac{\partial^2 W}{\partial I_2^2}I_{2,b} + 2\frac{\partial^2 W}{\partial I_2 \partial I_3}I_{3,b}\\
\text{and}\quad \gamma_{,b} &= 2\frac{\partial^2 W}{\partial I_1 \partial I_3}I_{1,b} + 2\frac{\partial^2 W}{\partial I_2 \partial I_3}I_{2,b} + 2\frac{\partial^2 W}{\partial I_3^2}I_{3,b}.
\end{aligned}\right\} \tag{4.4}$$

Substituting the above relations into the equilibrium equations one obtains

$$\left(-\frac{I_{3,b}}{2I_3}b^{ab}+b^{ab}{}_{,b}\right)\frac{\partial W}{\partial I_1}+\left\{-\frac{I_{3,b}}{2I_3}\left(I_2\delta^{ab}-I_3c^{ab}\right)+I_{2,b}\delta^{ab}-I_{3,b}c^{ab}-I_3c^{ab}{}_{,b}\right\}\frac{\partial W}{\partial I_2}$$
$$+\frac{1}{2}I_{3,b}\delta^{ab}\frac{\partial W}{\partial I_3}+b^{ab}I_{1,b}\frac{\partial^2 W}{\partial I_1^2}+I_{2,b}\left(I_2\delta^{ab}-I_3c^{ab}\right)\frac{\partial^2 W}{\partial I_2^2}+I_3I_{3,b}\delta^{ab}\frac{\partial^2 W}{\partial I_3^2}$$
$$+\{I_{1,b}(I_2\delta^{ab}-I_3c^{ab})+I_{2,b}b^{ab}\}\frac{\partial^2 W}{\partial I_1\partial I_2}+(b^{ab}I_{3,b}+\delta^{ab}I_{1,b}I_3)\frac{\partial^2 W}{\partial I_1\partial I_3}$$
$$+\{I_{3,b}(I_2\delta^{ab}-I_3c^{ab})+I_3I_{2,b}\delta^{ab}\}\frac{\partial^2 W}{\partial I_2\partial I_3}=0. \tag{4.5}$$

The above equations must be satisfied for any choice of the energy function W and hence the partial derivatives of W can vary independently. This implies that the coefficients of the different partial derivatives must vanish separately, which leads to the following system of equations:

$$-\frac{I_{3,b}}{2I_3}b^{ab}+b^{ab}{}_{,b}=0, \tag{4.6}$$

$$-\frac{I_{3,b}}{2I_3}(I_2\delta^{ab}-I_3c^{ab})+I_{2,b}\delta^{ab}-I_{3,b}c^{ab}-I_3c^{ab}{}_{,b}=0, \tag{4.7}$$

$$I_{3,b}\delta^{ab}=0, \tag{4.8}$$

$$b^{ab}I_{1,b}=0, \tag{4.9}$$

$$I_{2,b}(I_2\delta^{ab}-I_3c^{ab})=0, \tag{4.10}$$

$$I_3I_{3,b}\delta^{ab}=0, \tag{4.11}$$

$$I_{1,b}(I_2\delta^{ab}-I_3c^{ab})+I_{2,b}b^{ab}=0, \tag{4.12}$$

$$b^{ab}I_{3,b}+\delta^{ab}I_{1,b}I_3=0 \tag{4.13}$$

and

$$I_{3,b}(I_2\delta^{ab}-I_3c^{ab})+I_3I_{2,b}\delta^{ab}=0. \tag{4.14}$$

From (4.8) and (4.12)–(4.14), we conclude that I_1, I_2 and I_3 are constant. Equations (4.6) and (4.7) are satisfied if $b^{ab}{}_{,b}=c^{ab}{}_{,b}=0$. The remaining three equations are trivially satisfied. Therefore, we obtain Erisksen's conditions:

$$I_1, I_2, I_3 \text{ are constants,} \quad \text{and} \quad b^{ab}{}_{,b}=c^{ab}{}_{,b}=0 \tag{4.15}$$

and we refer to the conditions on the tensors **b** and **c** as the **b**-condition and **c**-condition, respectively. We note that, if **F** is a solution to the system of nonlinear PDEs (4.15), $k\mathbf{F}$ is also a solution for any constant k since

$$(I_1, I_2, I_3)\to(k^2I_1, k^4I_2, k^6I_3) \quad \text{and} \quad (\mathbf{b},\mathbf{c})\to(k^2\mathbf{b}, k^{-2}\mathbf{c}). \tag{4.16}$$

The constraints (4.15) are identical to those obtained by Ericksen [1]. However, in this study the five scalar and tensorial quantities explicitly depend on **G** and hence on the eigenstrains.

Interestingly, Ericksen fully understood the geometric content of the problem as he used the fact that for a flat reference configuration the compatibility equations can be written as the vanishing of the curvature tensor of the left Cauchy–Green deformation tensor. This is a consequence of the fact that the curvature tensor transforms naturally under push-forward by the deformation mapping. Note that $\mathbf{c}^\flat=\varphi_*\mathbf{G}$, and hence

$$\mathcal{R}(\mathbf{c}^\flat)=\mathcal{R}(\varphi_*\mathbf{G})=\varphi_*\mathcal{R}(\mathbf{G}). \tag{4.17}$$

If the reference configuration is flat $\mathcal{R}(\mathbf{G})=\mathbf{0}$ and hence $\mathcal{R}(\mathbf{c}^\flat)=\mathbf{0}$, the condition used by Ericksen. Then, the only universal deformations for compressible and isotropic solids are homogeneous deformations. When both the reference and current configurations are Euclidean, a homogeneous deformation is a constant linear map $\mathbf{A}$ and is written as

$$\mathbf{x} = \mathbf{A}\mathbf{X} + \mathbf{c}, \tag{4.18}$$

where $\mathbf{c}$ is a constant vector.

In generalizing this problem to pre-strained solids, there are two important points to consider. First, we have $\boldsymbol{\mathcal{R}}(\mathbf{G}) \neq \mathbf{0}$, in general, and a different condition for compatibility should be used. Second, the material manifold is non-trivial and there is no notion of a linear map between two manifolds, in general [27]. Indeed, in the presence of eigenstrains and when coordinate charts $\{X^A\}$ and $\{x^a\}$ are chosen for $\mathcal{B}$ and $\mathcal{S}$, respectively, a deformation gradient with constant coefficients $F^a{}_A$ does not lead to a deformation gradient that is constant from point to point. In particular, the condition that all $F^a{}_A$'s are constant does not guarantee $b^{ab}{}_{,b} = 0$. However, the notion of homogeneity can be defined intrinsically using covariant differentiation but it explicitly depends on the Levi–Civita connection and hence $\mathbf{G}$:

Definition 4.1. A deformation is said to be *covariantly homogeneous* if $\nabla^{\mathbf{G}}\mathbf{F} = \mathbf{0}$, that is, the covariant derivative of its deformation gradient vanishes identically.

We can now prove the following theorem:

Theorem 4.2. *A deformation in a compressible isotropic elastic solid with finite eigenstrains is universal if and only if it is covariantly homogeneous.*

Proof. When the ambient space is Euclidean and the material manifold is non-flat, the compatibility equations can be written as [36]: $d\mathbf{F} = \mathbf{0}$, where the deformation gradient $\mathbf{F}$ is viewed as a vector-valued 1-form, and d is the exterior derivative. In components, these compatibility equations read

$$\frac{\partial F^a{}_A}{\partial X^B} = \frac{\partial F^a{}_B}{\partial X^A}, \quad A < B, \;\; A, B = 1, 2, 3, \;\; a = 1, 2, 3. \tag{4.19}$$

These equations can be written in terms of covariant derivatives of the deformation gradient $\nabla^{\mathbf{G}}\mathbf{F}$. Explicitly, the covariant derivates are [24]

$$F^a{}_{A|B} = \frac{\partial F^a{}_A}{\partial X^B} - \Gamma^C{}_{AB} F^a{}_C + \gamma^a{}_{bc} F^b{}_A F^c{}_B. \tag{4.20}$$

Using the fact that the Levi–Civita connection is symmetric, i.e. $\Gamma^C{}_{AB} = \Gamma^C{}_{BA}$, the compatibility equations can be written

$$F^a{}_{A|B} = F^a{}_{B|A}. \tag{4.21}$$

If we choose a Cartesian coordinate chart for the Euclidean ambient space $\gamma^a{}_{bc} = 0$ and hence

$$F^a{}_{A|B} = \frac{\partial F^a{}_A}{\partial X^B} - \Gamma^C{}_{AB} F^a{}_C. \tag{4.22}$$

The $\mathbf{b}$-condition can then be rewritten as $b^{ab}{}_{,b} = (F^{-1})^A{}_b b^{ab}{}_{,A} = 0$. Note that, when Cartesian coordinates are used for the ambient space, $b^{ab}{}_{,A} = b^{ab}{}_{|A}$ and hence $b^{ab}{}_{,b} = 0$ can be written as $b^{ab}{}_{,b} = (F^{-1})^A{}_b b^{ab}{}_{|A} = 0$. In a Riemannian manifold, the metric tensor is covariantly constant. Therefore, we have

$$b^{ab}{}_{|A} = (F^a{}_M F^b{}_N G^{MN})_{|A} = F^a{}_{M|A} F^b{}_N G^{MN} + F^a{}_M F^b{}_{N|A} G^{MN} \tag{4.23}$$

and

$$b^{ab}{}_{,b} = F^a{}_{M|N} G^{MN} + F^a{}_M (F^{-1})^A{}_b F^b{}_{N|A} G^{MN} = 0. \tag{4.24}$$

Using the relation $F^a{}_A (F^{-1})^A{}_b = \delta^a{}_b$, one can show that

$$(F^{-1})^A{}_{a|B} = -(F^{-1})^A{}_b F^b{}_{C|B} (F^{-1})^C{}_a. \tag{4.25}$$

Using the $\mathbf{b}$-condition and $b^{ab} c_{bd} = \delta^a_d$ one obtains

$$c_{mn|A} = -c_{mp} c_{nq} b^{pq}{}_{|A} \quad \text{and} \quad c^{ab}{}_{|A} = -\delta^{am} \delta^{bn} c_{mp} c_{nq} b^{pq}{}_{|A}. \tag{4.26}$$

Therefore, the **c**-condition reads $\delta^{am}\delta^{bn}(F^{-1})^A{}_b c_{mp} c_{nq} b^{pq}{}_{|A} = 0$, which can be rewritten as

$$(F^{-1})^A{}_b[(F^{-1})^M{}_m(F^{-1})^N{}_n]_{|A} G_{MN}\delta^{am}\delta^{bn} = 0. \tag{4.27}$$

Hence, using (4.25), the **c**-condition can be expressed in terms of the covariant derivative of the deformation gradient as

$$(F^{-1})^A{}_b(F^{-1})_{Np}[(F^{-1})^{Nb}(F^{-1})^{Ma} + (F^{-1})^{Na}(F^{-1})^{Mb}]F^p{}_{M\,|\,A} = 0. \tag{4.28}$$

Next, we express the condition on the principal invariants in terms of covariant derivatives. We recall that the principal invariants of $\mathbf{b}^\sharp$ are

$$I_1 = \mathrm{tr}\,\mathbf{b}^\sharp = b^{ab} g_{ab}, \tag{4.29}$$

$$I_2 = \frac{1}{2}[I_1^2 - \mathrm{tr}((\mathbf{b}^\sharp)^2)] = \frac{1}{2}(I_1^2 - b^a{}_m b^{mb} g_{ab}) \tag{4.30}$$

and

$$I_3 = J^2 = \frac{\det \mathbf{g}}{\det \mathbf{G}}(\det \mathbf{F})^2. \tag{4.31}$$

Since I_1, I_2 and I_3 are constant, their covariant derivatives vanish identically $I_{1\,|\,A} = I_{2\,|\,A} = I_{3\,|\,A} = 0$. We consider these three conditions in turn. First,

$$0 = I_{1\,|\,A} = F^a{}_{M\,|\,A}F^b{}_N G^{MN}\delta_{ab} + F^a{}_M F^b{}_{N\,|\,A} G^{MN}\delta_{ab} = 2F^a{}_{M\,|\,A}F^b{}_N G^{MN}\delta_{ab}. \tag{4.32}$$

Thus, $F^a{}_{M\,|\,A}F^b{}_N G^{MN}\delta_{ab} = 0$, which can written as $F^M{}_a F^a{}_{M\,|\,A} = 0$. Second, $I_{2\,|\,A} = 0$ implies that $(b^a{}_m b^{mb} g_{ab})_{|A} = 0$ and, hence, $(F^a{}_A F^b{}_B F^m{}_M F^n{}_N)_{|C} G^{AM} G^{BN}\delta_{ab}\delta_{mn} = 0$. This is simplified to read

$$(F_m{}^M F^{mN} + F_m{}^N F^{mM})F_{nN}F^n{}_{M\,|\,A} = 0. \tag{4.33}$$

Third, we use Piola's identity [24], $[J(F^{-1})^A{}_a]_{|A} = 0$, and the fact that I_3 is a constant to obtain $(F^{-1})^A{}_{a\,|\,A} = 0$. Using (4.25) and compatibility, we can rewrite this last identity as $(F^{-1})^B{}_b F^b{}_{A\,|\,B} = (F^{-1})^B{}_b F^b{}_{B\,|\,A} = 0$. Using this last relation, the **b**-condition (4.24) is simplified to read

$$F^a{}_{M\,|\,N} G^{MN} = 0. \tag{4.34}$$

Gathering both Ericksen's and the compatibility conditions, we have the following set of equations for the unknown $\nabla^{\mathbf{G}}\mathbf{F}$:

$$\text{compatibility}: \quad F^a{}_{A\,|\,B} = F^a{}_{B|A}, \tag{4.35}$$

$$I_1 = \text{constant}: \quad F^M{}_n F^n{}_{M\,|\,A} = 0, \tag{4.36}$$

$$I_2 = \text{constant}: \quad (F^M{}_m F^{mN} + F^N{}_m F^{mM})F_{nN}F^n{}_{M\,|\,A} = 0, \tag{4.37}$$

$$I_3 = \text{constant}: \quad (F^{-1})^M{}_n F^n{}_{M\,|\,A} = 0, \tag{4.38}$$

$$b^{ab}{}_{,b} = 0: \quad G^{MN}F^a{}_{M|N} = 0 \tag{4.39}$$

and

$$c^{ab}{}_{,b} = 0: \quad (F^{-1})^A{}_b(F^{-1})_{Np}[(F^{-1})^{Nb}(F^{-1})^{Ma} + (F^{-1})^{Na}(F^{-1})^{Mb}]F^p{}_{M\,|\,A} = 0. \tag{4.40}$$

The sufficiency part of the theorem follows directly: if $\nabla^{\mathbf{G}}\mathbf{F} = \mathbf{0}$, all the covariant derivatives vanish, then these equations are identically satisfied and $\mathbf{F}$ is the deformation gradient of a universal deformation.

The necessity part of the theorem consists in showing that $\nabla^{\mathbf{G}}\mathbf{F}=\mathbf{0}$ is the only solution. Since I_1 is a constant both its covariant derivative (4.32) and its second covariant derivative vanish (and hence exist). Therefore, we have

$$0=(F^a{}_{M|A}F^b{}_N G^{MN}\delta_{ab})_{|B}=F^a{}_{M|AB}F^b{}_N G^{MN}\delta_{ab}+F^a{}_{M|A}F^b{}_{N|B}G^{MN}\delta_{ab}. \quad (4.41)$$

Taking the trace of the above tensor one obtains

$$F^a{}_{M|AB}F^b{}_N G^{MN}\delta_{ab}G^{AB}+F^a{}_{M|A}F^b{}_{N|B}G^{MN}\delta_{ab}G^{AB}=0, \quad (4.42)$$

where we recognize the second term on the left-hand side as $|\nabla^{\mathbf{G}}\mathbf{F}|$, so that

$$F^a{}_{M|AB}F^b{}_N G^{MN}\delta_{ab}G^{AB}+|\nabla^{\mathbf{G}}\mathbf{F}|=0. \quad (4.43)$$

It remains to prove that the first term vanishes identically. Using the compatibility equations and the definition of the curvature tensor one can write

$$F^a{}_{M|AB}=F^a{}_{A|MB}=F^a{}_{A|BM}+\mathcal{R}^D{}_{MBA}F^a{}_D, \quad (4.44)$$

where $\mathcal{R}$ is the material Riemannian curvature tensor and we have used the fact that, as the ambient space is Euclidean and $\mathbf{F}$ is an $\mathbb{R}^n$-valued 1-form, Ricci's identity (2.6) can be used to express the commutator of covariant derivatives of the deformation gradient in terms of the curvature tensor. Therefore

$$F^a{}_{M|AB}G^{AB}=F^a{}_{A|BM}G^{AB}+G^{AB}\mathcal{R}^D{}_{MBA}F^a{}_D=(F^a{}_{A|B}G^{AB})_{|M}, \quad (4.45)$$

where use was made of the facts that $\mathbf{G}$ is covariantly constant and $\mathcal{R}^D{}_{MBA}=-\mathcal{R}^D{}_{MAB}$. Now using the $\mathbf{b}$-condition (4.39) in the above equation, we conclude that $F^a{}_{M|AB}G^{AB}=0$. Hence, the first term on the left-hand side of (4.43) is identically zero and we have shown that

$$F^a{}_{M|A}F^b{}_{N|B}G^{MN}\delta_{ab}G^{AB}=|\nabla^{\mathbf{G}}\mathbf{F}|^2=0, \quad (4.46)$$

which implies $\nabla^{\mathbf{G}}\mathbf{F}=\mathbf{0}$. ■

In the absence of eigenstrains, the material metric is flat and a covariantly homogeneous deformation is simply a classical homogeneous deformation. Therefore, we have:

Corollary 4.3 (Ericksen's theorem). *In a compressible isotropic elastic solid with zero eigenstrain, a deformation is universal if and only if it is homogeneous.*

5. Universal eigenstrains are zero-stress

Next, we wish to characterize the nature of eigenstrains so that the conditions $\nabla^{\mathbf{G}}\mathbf{F}=\mathbf{0}$ for non-trivial $\mathbf{G}$ can have a solution for $\mathbf{F}$. If $\nabla^{\mathbf{G}}\mathbf{F}=\mathbf{0}$, then in a local coordinate chart $F^a{}_{A|B}=0$ and hence $F^{aA}{}_{|B}=0$, where $F^{aA}=G^{AB}F^a{}_B$. This implies that the Riemannian manifold $(\mathcal{B},\mathbf{G})$ admits a global covariantly constant vector field. Using Ricci's identity (2.5) we have $0=F^{aA}{}_{|BC}-F^{aA}{}_{|CB}=\mathcal{R}^A{}_{BCD}F^{aD}$. Therefore

$$\mathcal{R}^A{}_{ACD}F^{aD}=R_{CD}F^{aD}=0. \quad (5.1)$$

This means that Ricci curvature has a zero eigenvalue, which is an integrability condition for $\nabla^{\mathbf{G}}\mathbf{F}=\mathbf{0}$ [37].

In dimension 2, we can write

$$R_{CD}F^{aD}=\tfrac{1}{2}RG_{CD}F^{aD}=0. \quad (5.2)$$

Knowing that the metric is positive-definite one concludes that $R=0$ and hence the material manifold is flat. In other words, in dimension 2 universal eigenstrains are zero-stress and universal deformations are covariantly homogeneous. Therefore, we have proved the following theorem.

Theorem 5.1. *In dimension* 2, *all universal eigenstrains are zero-stress.*

The same result holds for three-dimensional bodies:

Theorem 5.2. *In dimension* 3, *all universal eigenstrains are zero-stress.*

Proof. We prove the theorem in three steps.

Step 1: If the material manifold is not a symmetric Riemannian manifold, Berger [29] proved that the only possible holonomy group for a real 3-manifold is $SO(3)$. This means that any global covariantly constant vector field is $SO(3)$-invariant, which implies no non-trivial covariantly constant vector field. Therefore, the material manifold $(\mathcal{B}, \mathbf{G})$ must be a symmetric Riemannian manifold in order to have a non-trivial homogeneous deformation field. $(\mathcal{B}, \mathbf{G})$ is either a reducible or irreducible Riemannian manifold.

Step 2: If $(\mathcal{B}, \mathbf{G})$ is irreducible, then it must be an Einstein manifold [30,31]. Since the Ricci curvature has a zero eigenvalue, we conclude that the Ricci curvature vanishes and hence the material manifold is flat.

Step 3: If the three-dimensional Riemannian manifold $(\mathcal{B}, \mathbf{G})$ is reducible, the only non-trivial splittings of $\mathcal{B}$ would be either $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2$, where $\dim \mathcal{B}_1 = 1$ and $\dim \mathcal{B}_2 = 2$, or $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3$, where $\dim \mathcal{B}_1 = \dim \mathcal{B}_2 = \dim \mathcal{B}_3 = 1$. In the first case, $\mathcal{B}_1$ is flat and, since $(\mathcal{B}, \mathbf{G})$ is homogeneous, we conclude that $\nabla^{\mathbf{G}_2} \mathbf{R}_2 = \mathbf{0}$. Therefore, $\mathbf{R}_2 = k\mathbf{G}_2$, where k is a constant. However, knowing that $(\mathcal{B}, \mathbf{G})$ is homogeneous, from the Bianchi identities we know that its scalar curvature vanishes. But $(\mathcal{B}, \mathbf{G})$ and $(\mathcal{B}_2, \mathbf{G}_2)$ have the same scalar curvatures and hence $k = 0$. In the second case, $(\mathcal{B}, \mathbf{G})$ is trivially flat.

This completes the proof of the flatness of the material manifold, which in turn implies that the universal eigenstrains are zero-stress. ∎

6. Concluding remarks

In this paper, we studied the universal deformations in compressible elastic bodies with finite eigenstrains that are isotropic in the absence of eigenstrains. We extended the classic notion of homogeneity to the geometric notion of covariant homogeneity based on covariant differentiation. Since covariant differentiation on the material manifold inherently depends on the distribution of eigenstrains, covariantly homogeneous deformations also depend on the distribution of eigenstrains. We showed that, in dimension 2, the material manifold has to be flat and hence eigenstrains must be zero-stress. In dimension 3, we showed that the material manifold has to be a Riemannian symmetric space. Assuming that the body is simply-connected, we showed that the material manifold must be flat for a covariantly homogeneous deformation to exist. In summary, in dimensions 2 and 3 for a simply-connected body, the only universal eigenstrains are the zero-stress eigenstrains and all universal deformations are covariantly homogeneous. We end with a couple of suggestions as possible extensions. The first natural extension is to generalize the main theorem to non-simply-connected bodies. Unfortunately, there are no theorems for classification of holonomy groups in this case and no direct results can be obtained. The next natural problem, following Ericksen's footsteps, is to consider the problem of determining all universal solutions and universal eigenstrains for incompressible anelastic solids. As for the elastic case, this problem is considerably more difficult but most of the apparatus developed here can be used to tackle this important problem.

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